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SEVERAL MODELS OF SIX DEGREE OF FREEDOM EQUATIONS OF MOTION FOR A BALLISTIC MISSILE

Prepared by

CHRYSLER CORPORATION MISSILE DIVISION

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SEVERAL MODELS OF SIX DEGREE OF FREEDOM EQUATIONS OF MOTION FOR A BALLISTIC MISSILE

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ABSTRACT

Presented in this report are total and perturbation six degree of freedom equations of motion and their derivations. Magnus force and moment were considered negligible and not included in this derivation, however, when desired they can be incorporated in the existing models. Definitions of control force, gravity, thrust and damping terms are presented in general form so that various control schemes and particular functions of the terms can be utilized without extensive modification to the equations of motion. These equations may be programmed on a large scale digital or analog computer.

The majority of the work presented herein was developed for previous programs and adapted for use in the Penetration Aids Program under Air Force Contract AF04(694)-25.

OBJECT:

The object of this report is to derive and present six-degree-of-freedom equations of motion for a rigid ballistic missile.

INTRODUCTION:

Flight mechanics problems involving non-spinning vehicles can be solved, in general, with dynamical equations which are independent of the vehicle's point mass trajectory. In the spinning body problem, however, the orientation of the vehicle and its motion are mutually related as a result of aerodynamic and inertial coupling and, therefore, must be studied with a six-degree-of-freedom model. This report presents the derivation of several six-degree-of-freedom models for a general problem. It has been assumed in this derivation that the earth is spherical and non-rotating and that body spin rates are small enough to neglect Magnus forces.

The c.g. fixed part of the motion of the body is expressed non-holonomically, in equations (3), as dynamical equilibrium about any three right-handed, orthogonal cartesian axes fixed in the body. The coordinates, p, q, and r, of equations (3) are known classically as the rectilinear, but non-holonomic coordinates of rigid body rotation dynamics. These coordinates can be expressed in terms of holonomic, but curvilinear coordinates such as Eulerian Angles. Equations (4) represent transformation of p, q, and r to the particular Eulerian Angles ψ, θ, and ϕ. By means of transformation (4), the dynamical equilibrium expressed non-holonomically by equations (3) can be reexpressed holonomically as it is in equations (5). The dynamical equations for body rotation are non-holonomic in p, q, and r because the inertia terms in (3) are only once integrable without the adjunction of equations (4).

Immediately subsequent to equations (4), in this report, the question of what coordinates to use is decided. Two approaches, designated moment derivation A and moment derivation B, are adopted. The starting points for these are equations (5) for A and equations (3) for B. In both "derivations", the applied torque terms are expressed in terms of Eulerian Angles, ψ , θ , and ϕ . Only in A, however, are the inertia terms also expressed in ψ , θ , and ϕ . An interim result in A is equations (18) which are a direct combination of the inertia terms of equations (5) with the applied torque terms of equations (17). Equations (19) through (25) are concerned with the translational equations of motion of the body in the space-fixed coordinates, x, y, and z of translation and in the space-fixed coordinates ψ , θ , and ϕ of orientation. Equations (18) together with equations (25) constitute a complete set of holonomic, but non-linear, dynamic, six-degree-of-freedom equations of motion of the rigid body in the all-space-fixed coordinates x, y, z, ψ , θ , and ϕ . Because the individual rotation equations of this system represent dynamical equilibrium of components along body-fixed axes, only constant-valued second and product moments of inertia occur. Equations (18) plus equations (25), as a system, are incomplete in that flexible-body degrees-of-freedom are excluded and are inexact in that (1) magnus forces are neglected and (2) a spherical, non-rotating earth is assumed.

Further development under moment derivation A is desirable because bodies either symmetric about the roll axis or nearly symmetric about the roll axis are a frequent object of dynamical analysis. In the translational degrees-of-freedom, bodies with roll symmetry can be governed no more simply than by equations (such as equations (25) which govern unsymmetrical bodies. In the rotational degrees-of-freedom, however, a form simpler than equations (18) is possible for the roll symmetric body. Such symplification obtains by the following reasoning. Instead of equations (18) which represent dynamic equilibrium of components of applied torque along body-fixed axes with components of the applicable, reactive, inertia torques along the same axes, equations still based on ψ , θ , and ϕ but specifying dynamical equilibrium of corresponding components along any suitably limited set of three axes can be used with equal efficacy to fix the rotational behavior of the unsymmetrical body. In general, a (pre-solution) choice of component axes fixed-in-space tends to produce an inertially uncoupled set of equations, but also leads to variable second and product moment of inertia coefficients. Constant-inertia-coefficient equations applicable to the unsymmetric body require a choice of component axes, such as that choice resulting in equations (18), which follow the body in yaw, pitch, and roll. For bodies symmetrical to the roll axis, a choice of component axes which follow the body only in yaw and pitch gives constant-inertia-coefficient equations with a form less inertially coupled than equations (18). The x', y', and z' axes system follows the body in yaw, pitch, and roll while the x_1, y_1, and z_1 axes system follows the body only in yaw and pitch. Equations (26) express, for any static moment vector, components along the instantaneous x_1, y_1, and z_1 axes in terms of components along the instantaneous x', y', and z' axes. Application of equations (26) to the inertia torque terms and to the applied torque terms of equations (18) yields equation of dynamic equilibrium in rotation about the x_1 , y_1 , and z_1 axes. These equations are valid for the unsymmetric body. Subsequent specialization to the roll symmetric body is readily accomplished by setting every second moment of inertia about any axis perpendicular to the roll axis equal to a single value, I, and by setting all product moments of inertia equal to zero. The result is equations (27) which are exact for the roll symmetric body in the same sense that equations (18) are exact for the unsymmetric body. Equations (25) together with equations (27) constitute a complete set of holonomic, but non-linear, dynamic, six-degree-of-freedom equations of motion of the roll-symmetric, rigid body in the all-space-fixed coordinates x, y, z, ψ , θ , and ϕ . Because the individual rotation equations of this system represent dynamical equilibrium of components along axes which follow the body in yaw and pitch, and because they are based on the assumption of roll-symmetry of the body, only constant valued second moments of inertia occur and product moments of inertia do not occur at all.

Continued development under moment derivation A consists of forming perturbation equations of motion from the system composed of equations (25) together with equations (27). This is accomplished by subtracting equations corresponding to a standard, zero angle of attack trajectory from equations (25) and (27). For this standard trajectory, equations (25) and (27) reduce to equations (28). When equations (28) are subtracted, then, from equations (25) and (27), the result is equations (29) which are the desired perturbation equations of motion. For applications to roll-symmetric bodies and where the yaw angle, ψ , and the miscellaneous angles β_p and β_Y are small enough to linearize, equations (29) reduce to the (approximate) equations (30). Equations (31) through (34) are concerned with introducing the parameters α_p and α_Y and with further linearization of equations (30). Equations (34) are the final result under moment derivation A and represent approximate equations of motion for the roll symmetric body. Assumptions incorporated in equations (34) and not incorporated in equations (30) are that ($\theta - \theta_s$), $\frac{\dot{x} - \dot{x}_s}{V}$, $\frac{\dot{y} - \dot{y}_s}{V}$, and $\frac{\dot{z} - \dot{z}_s}{V}$ are small enough to linearize.

Under moment derivation B, only the rotation equations are treated. The translational equations are the same for B as for A. Equations (35) are a set of three, non-holonomic, six-degree-of-freedom equations for dynamic equilibrium of torque components along the x', y', and z' axes system in nine coordinates, p, q, r, x, y, z, ψ , θ , and ϕ . Together with the three translational equations (25), and with the three "constraint" equations (4), equations (35) form a complete set of nine, non-holonomic, six-degree-of-freedom, equations of motion applicable to the general, unsymmetric, rigid body. Equation (35) are formed by setting the applied torque terms of equations (17) equal to the reactive, inertia torque terms of equations (3) and represent dynamic, rotational equilibrium about axes which follow the body in yaw, pitch, and roll.

For the roll symmetric body in motion linearizable with respect to ψ, β_p, β_r, $(\Theta - \Theta_s)$, $\frac{\dot{x} - \dot{x}_s}{V}$, $\frac{\dot{y} - \dot{y}_s}{V}$, and $\frac{\dot{z} - \dot{z}_s}{V}$, equations (36) supplant equations (35) for expressing rotational equilibrium and equations (34a), (34b), and (34c) supplant equations (25) for translational equilibrium. Equations (36) differ from 34d, 34e, and 34f, not only because of the use of p, q, and r in expressing the reactive inertia terms, but also because equations (36) represent dynamic equilibrium about axes which follow the body in yaw, pitch, and roll while equations 34d, 34e, and 34f represent dynamic equilibrium about axes which follow the body only in pitch and yaw.

CONCLUSIONS:

It is concluded that this report presents useful forms for the following sets of equations.

Eqns. (18) Total rotational equations in terms of Eulerian angles.

Eqns. (25) Total translational equations along space fixed axes.

Eqns. (29) Simplified six-degree-of-freedom perturbation equations.

Eqns. (34) Linearized six-degree-of-freedom perturbation equations.

Eqns. (35) Total body moment equations in terms of body angular rates.

Eqns. (36) Linearized body moment perturbation equations in terms of body angular rates.

Derivation of the Equations of Motion

The translational equations of motion are written with respect to a spaced fixed axes system in order to measure true accelerations in fixed space to which Newtonian laws apply. The moment equations are written relative to body fixed axes with origin at the center of gravity and parallel to the axes of symmetry of the body. Body fixed axes eliminate the need for considering the moments of inertia as a function of vehicle orientation. The thrust, control force, gravity, and damping definitions are general since they will be uniquely defined for each problem. The moment or rotational equations of motion are considered first.

Section I - Rotational Equations of Motion

From reference 1 and 2 the moments of momentum of a missile about its body fixed axes system are defined as:

Moments of Momentum

(1)
$$
\begin{aligned}
h_{X'} &= p I_{X'} - q J_{X'Y'} - r J_{X'Z'} \\
h_{Y'} &= q I_{Y'} - p J_{X'Y'} - r J_{Y'Z'} \\
h_{Z'} &= r I_{Z'} - p J_{Z'X'} - q J_{Y'Z'}
\end{aligned}
$$

From figure 1, the moments about the body fixed axes, defined as the rate of change of moment of momentum about those axes, are expressed as:

(2)
$$
\begin{aligned}
M_{Roll}\ (\text{Body}) &= \frac{dh_{X'}}{dt} - h_{Y'} r + h_{Z'} q \\
M_{Pitch}\ (\text{Body}) &= \frac{dh_{Y'}}{dt} - h_{Z'} p + h_{X'} r \\
M_{Yaw}\ (\text{Body}) &= \frac{dh_{Z'}}{dt} - h_{X'} q + h_{Y'} p
\end{aligned}
$$

Substituting equations (1) into equations (2), presents the body moment equations in the form:

$$
\begin{aligned}
M_{Roll} &= \dot{p} I_{x'} + (rp - \dot{q}) J_{x'y'} - (\dot{r} + pq) J_{z'x'} + (r^2 - q^2) J_{y'z'} + qr(I_z - I_y) \\
(3) \quad M_{Pitch} &= \dot{q} I_{y'} + (pq - \dot{r}) J_{y'z'} - (\dot{p} + qr) J_{x'y'} + (p^2 - r^2) J_{z'x'} + rp(I_x - I_z) \\
M_{Yaw} &= \dot{r} I_{z'} + (qr - \dot{p}) J_{x'z'} - (\dot{q} + rp) J_{y'z'} + (q^2 - p^2) J_{x'y'} + pq(I_{y'} - I_{x'})
\end{aligned}
$$

In these equations the terms containing the time derivatives of moment of inertia have been neglected.

The angular rates about the body axes in terms of Eulerian angles are shown from figure 2 to be:

$$
\begin{aligned}
p &= \dot{\phi} - \dot{\psi} \sin\theta \\
(4) \quad q &= \dot{\theta} \cos\phi + \dot{\psi} \cos\theta \sin\phi \\
r &= \dot{\psi} \cos\theta \cos\phi - \dot{\theta} \sin\phi
\end{aligned}
$$

At this point in the derivation a choice must be made to proceed by either expressing body moment in terms of Eulerian angles and their rates or in terms of body angular rates and accelerations. Both are presented, however, the former is taken first in this report. The derivations will be labeled A and B for their respective occasion in the report.

Moment Derivation A

Substituting the values of p, q, and r as defined above, and their derivatives into equations 3, the body moment equations becomes:

$$
\begin{aligned}
\sum M_{Roll} = \; & I_{x'}(\ddot{\phi} - \ddot{\psi}\sin\theta - \dot{\psi}\dot{\theta}\cos\theta) \\
& + (I_{z'} - I_{y'})\{\dot{\theta}\dot{\psi}(\cos^2\phi - \sin^2\phi)\cos\theta + \dot{\psi}^2\cos^2\theta\sin\phi\cos\phi - \dot{\theta}^2\cos\phi\sin\phi\} \\
& - J_{x'z'}(\ddot{\psi}\cos\theta\cos\phi - 2\dot{\psi}\dot{\theta}\sin\theta\cos\phi - \ddot{\theta}\sin\phi - \dot{\psi}^2\cos\theta\sin\theta\sin\phi) \\
& + J_{x'y'}(2\dot{\psi}\dot{\theta}\sin\theta\sin\phi - \dot{\psi}^2\cos\theta\sin\theta\cos\phi - \ddot{\theta}\cos\phi \\
& \qquad - \ddot{\psi}\cos\theta\sin\phi) \\
& + J_{y'z'}\{\dot{\psi}^2\cos^2\theta(\cos^2\phi - \sin^2\phi) - 4\dot{\theta}\dot{\psi}\cos\phi\cos\theta\sin\phi \\
& \qquad + \dot{\theta}^2(\sin^2\phi - \cos^2\phi)\}
\end{aligned} \tag{5a}
$$

$$
\begin{aligned}
\sum M_{Pitch} = \; & I_{y'}(\ddot{\theta}\cos\phi - \dot{\theta}\dot{\phi}\sin\phi + \ddot{\psi}\cos\theta\sin\phi - \dot{\psi}\dot{\theta}\sin\theta\sin\phi \\
& \qquad + \dot{\psi}\dot{\phi}\cos\theta\cos\phi) \\
& + (I_{x'} - I_{z'})(\dot{\phi}\dot{\psi}\cos\theta\cos\phi - \dot{\phi}\dot{\theta}\sin\phi - \dot{\psi}^2\sin\theta\cos\theta\cos\phi \\
& \qquad + \dot{\psi}\dot{\theta}\sin\theta\sin\phi) \\
& + J_{x'z'}(\dot{\phi}^2 - 2\dot{\phi}\dot{\psi}\sin\theta + \dot{\psi}^2\sin^2\theta - \dot{\psi}^2\cos^2\theta\cos^2\phi \\
& \qquad + 2\dot{\psi}\dot{\theta}\cos\theta\cos\phi\sin\phi - \dot{\theta}^2\sin^2\phi) \\
& - J_{x'y'}(\ddot{\phi} - \ddot{\psi}\sin\theta - 2\dot{\theta}\dot{\psi}\cos\theta\sin^2\phi - \dot{\theta}^2\sin\phi\cos\phi \\
& \qquad + \dot{\psi}^2\cos^2\theta\sin\phi\cos\phi) \\
& + J_{y'z'}(\ddot{\theta}\sin\phi + 2\dot{\theta}\dot{\phi}\cos\phi + 2\dot{\phi}\dot{\psi}\cos\theta\sin\phi \\
& \qquad - \dot{\psi}^2\sin\theta\cos\theta\sin\phi - \ddot{\psi}\cos\theta\cos\phi)
\end{aligned} \tag{5b}
$$

$$
\begin{aligned}
\sum M_{YAW} = \; & I_{z'}\left(\ddot{\psi}\cos\theta\cos\phi - \dot{\psi}\dot{\theta}\sin\theta\cos\phi - \dot{\phi}\dot{\psi}\cos\theta\sin\phi \right. \\
& \qquad \left. - \ddot{\theta}\sin\phi - \dot{\theta}\dot{\phi}\cos\phi\right) \\
& + (I_{Y'} - I_{X'})\left(\dot{\phi}\dot{\theta}\cos\phi + \dot{\phi}\dot{\psi}\cos\theta\sin\phi - \dot{\psi}\dot{\theta}\sin\theta\cos\phi \right. \\
& \qquad \left. - \dot{\psi}^2\cos\theta\sin\theta\sin\phi\right) \\
& - J_{X'Z'}\left(\ddot{\phi} - \ddot{\psi}\sin\theta + \dot{\theta}^2\cos\phi\sin\phi - 2\dot{\theta}\dot{\psi}\cos\theta\cos^2\phi \right. \\
& \qquad \left. - \dot{\psi}^2\cos^2\theta\sin\phi\cos\phi\right) \\
\text{(5c)} \qquad & - J_{YZ'}\left(\ddot{\theta}\cos\phi + 2\dot{\phi}\dot{\psi}\cos\theta\cos\phi - 2\dot{\theta}\dot{\phi}\sin\phi \right. \\
& \qquad \left. - \dot{\psi}^2\sin\theta\cos\theta\cos\phi + \ddot{\psi}\cos\theta\sin\phi\right) \\
& + J_{X'Y'}\left(\dot{\theta}^2\cos^2\phi + 2\dot{\theta}\dot{\psi}\cos\phi\cos\theta\sin\phi - \dot{\phi}^2 \right. \\
& \qquad \left. + 2\dot{\phi}\dot{\psi}\sin\theta + \dot{\psi}^2\{\cos^2\theta\sin^2\phi - \sin^2\theta\}\right)
\end{aligned}
$$

Applied Moment

The total applied moment about a body axis is comprised of an aerodynamic force moment, a control force moment, a thrust vector misalignment moment, and a damping moment. This can be written in equation form as:

$$
(6) \qquad M_T = M_{Aero.} + M_{C.F.} + M_{Th.} + M_D
$$

1. Aerodynamic Moment

Moment is expressed in vector analysis as:

$$
\bar{M}_{AERO} = \bar{r}_A \times \bar{F}
$$

Letting

$$
\bar{r}_A = \bar{i}' r_{X'} + \bar{j}' r_{Y'} + \bar{k}' r_{Z'}
$$

and

$$
\bar{F}_{AERO} = \bar{i}' F_{X'AERO} + \bar{j}' F_{Y'AERO} + \bar{k}' F_{Z'AERO}
$$

where $\vec{r}_A$ is the vector from the body C.G. to the application of the aerodynamic force and where i', j', and k' are unit vectors along the body x', y' and z' axes, respectively.

The above cross product has the following components:

$$M_{x'_{aero}} = \left[F_{Z'_{AERO}} r_{Y'} - F_{Y'_{AERO}} r_{Z'}\right]$$

(7) $$M_{y'_{aero}} = \left[F_{X'_{AERO}} r_{Z'} - F_{Z'_{AERO}} r_{X'}\right]$$

$$M_{z'_{aero}} = \left[F_{X'_{AERO}} r_{Y'} - F_{Y'_{AERO}} r_{X'}\right]$$

The aerodynamic force components as shown in figure 3 are:

$$F_{Z'_{AERO}} = \vec{F}_N \cdot \vec{k}' \qquad F_{Y'_{AERO}} = \vec{F}_N \cdot \vec{j}' \qquad F_{X'_{AERO}} = -D.$$

where:

$$\vec{F}_N = \frac{C_N Q A}{|V|\sin\alpha}\left[(\vec{V} \times \vec{i}') \times \vec{i}'\right] \quad \text{AND} \quad \vec{V} = \dot{X}\vec{i} + \dot{Y}\vec{j} + \dot{Z}\vec{K}$$

With the vector transformation on figure 2, velocity can be expressed along body axes so that F_N becomes:

$$\begin{aligned}\vec{F}_N = \frac{C_N Q A}{|V|\sin\alpha}\Big\{ -\vec{j}' \Big[& \dot{X}(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi) \\ & + \dot{Y}(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi) \\ & + \dot{Z}\sin\phi\cos\theta \Big] \\ -\vec{k}' \Big[& \dot{X}(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi) \\ & + \dot{Y}(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi) \\ & + \dot{Z}\cos\phi\cos\theta \Big] \Big\}\end{aligned}$$

The y' and z' components are:

(8a) $$\begin{aligned}\vec{F}_N \cdot \vec{j}' = \frac{C_N Q A}{|V|\sin\alpha}\Big[& -\dot{X}(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi) \\ & -\dot{Y}(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi) \\ & -\dot{Z}\sin\phi\cos\theta \Big]\end{aligned}$$

$$(8b)\qquad \vec{F}_N \cdot \vec{k}' = \frac{C_N Q A}{|V|\sin\alpha}\left[-\dot{X}(\cos\phi\sin\theta\cos\psi+\sin\phi\sin\psi) -\dot{Y}(\cos\phi\sin\theta\sin\psi-\sin\phi\cos\psi) -\dot{Z}\cos\phi\cos\theta\right]$$

The moment arm components are defined as:

$r_{x'} = -(CG-CP)$ where CG and CP are measured from an aft reference point

(9) $r_{y'} = \Delta CP_{y'}$

$r_{z'} = \Delta CP_{z'}$

$\Delta CP_{y'}$ and $\Delta CP_{z'}$ include the misalignment of the center of pressure and the center of gravity.

Substituting equations 8 and 9 into equations 7, presents the applied moment due to an aerodynamic force.

$$(10a)\qquad M_{x'_{aero}} = \frac{C_N Q A}{|\sin\alpha|}\left\{\Delta CP_{Y}\cdot\left[-\frac{\dot{X}}{|V|}(\cos\phi\sin\theta\cos\psi+\sin\phi\sin\psi) -\frac{\dot{Y}}{|V|}(\cos\phi\sin\theta\sin\psi-\sin\phi\cos\psi) -\frac{\dot{Z}}{|V|}\cos\phi\cos\theta\right] + \Delta CP_{Z}\cdot\left[\frac{\dot{X}}{|V|}(\sin\phi\sin\theta\cos\psi-\cos\phi\sin\psi) +\frac{\dot{Y}}{|V|}(\cos\phi\cos\psi+\sin\phi\sin\theta\sin\psi) +\frac{\dot{Z}}{|V|}\sin\phi\cos\theta\right]\right\}$$

$$M_{y'aero} = -D_0\left(\Delta CP_Z\right) \tag{10b}$$

$$+\frac{C_N QAD}{|\sin\alpha|}\left(\frac{CG-CP}{D}\right)\left[-\frac{\dot{X}}{|V|}\left(\cos\phi\sin\theta\cos\psi+\sin\phi\sin\psi\right)\right.$$

$$-\frac{\dot{Y}}{|V|}\left(\cos\phi\sin\theta\sin\psi-\sin\phi\cos\psi\right)$$

$$\left.-\frac{\dot{Z}}{|V|}\cos\phi\cos\theta\right]$$

$$M_{z'aero} = D_0\,\Delta CP_Y \tag{10c}$$

$$+\frac{C_N QAD}{|\sin\alpha|}\left(\frac{CG-CP}{D}\right)\left[\frac{\dot{X}}{|V|}\left(\sin\phi\sin\theta\cos\psi-\cos\phi\sin\psi\right)\right.$$

$$+\frac{\dot{Y}}{|V|}\left(\cos\phi\cos\psi+\sin\phi\sin\theta\sin\psi\right)$$

$$\left.+\frac{\dot{Z}}{|V|}\sin\phi\cos\theta\right]$$

2. **Control Force Moment**

This moment is the result of control forces obtained from either reaction jet or external movable fins applied as in diagram 1.

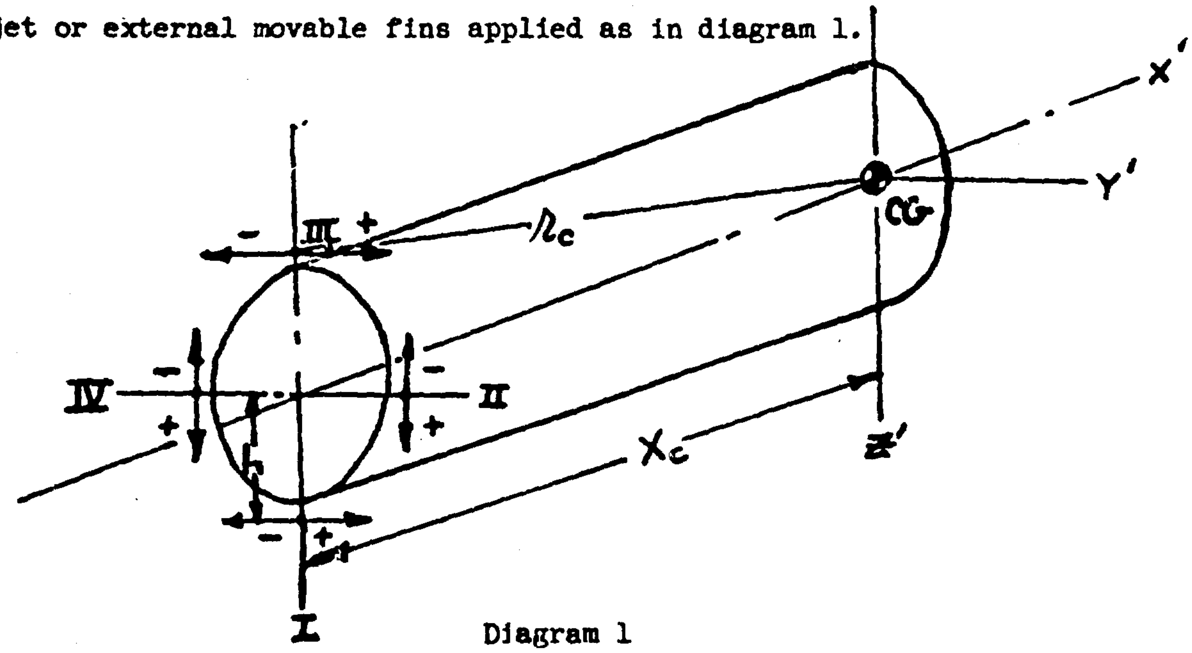

Diagram 1

Forces and moment arms are:

$$\vec{F}_I,\ \vec{F}_{III} = \pm\vec{j}'F_c \qquad \vec{r}_{I_c} = -\vec{i}'x_c + \vec{k}'h$$

$$\vec{F}_{II},\ \vec{F}_{IV} = \pm\vec{k}'F_c \tag{11}$$

$$\vec{r}_{II_c} = -\vec{i}'x_c + \vec{j}'h$$

$$\vec{r}_{III_c} = -\vec{i}'x_c - \vec{k}'h$$

$$\vec{r}_{IV_c} = -\vec{i}'x_c - \vec{j}'h$$

The control moment is defined as:

$$(12) \qquad \vec{M}_c = \sum \vec{r}_c \times \vec{F}_c$$

Evaluating each individual cross product gives:

$$(13) \quad \left.\begin{aligned}
\vec{r}_{I_c} \times \vec{F}_I &= F_I(-X_c)\vec{k}' - F_I h\,\vec{i}' \\
\vec{r}_{II_c} \times \vec{F}_{II} &= F_{II}\,h\,\vec{i}' + F_{II}X_c\,\vec{j}' \\
\vec{r}_{III_c} \times \vec{F}_{III} &= F_{III}\,h\,\vec{i}' - F_{III}X_c\,\vec{k}' \\
\vec{r}_{IV} \times \vec{F}_{IV} &= F_{IV}X_c\,\vec{j}' - F_{IV}h\,\vec{i}'
\end{aligned}\right\}$$

A summation of the individual control moments about the body axes produces the following equations:

$$(14a) \qquad M_{c_{roll}} = h\left(-F_I + F_{II} + F_{III} - F_{IV}\right)$$

$$(14b) \qquad M_{c_{pitch}} = X_c\left(F_{II} + F_{IV}\right)$$

$$(14c) \qquad M_{c_{yaw}} = -X_c\left(F_I + F_{III}\right)$$

3. <u>Misaligned Thrust Vector Moment</u>

The moment produced by a canted, displaced thrust vector can be derived as follows:

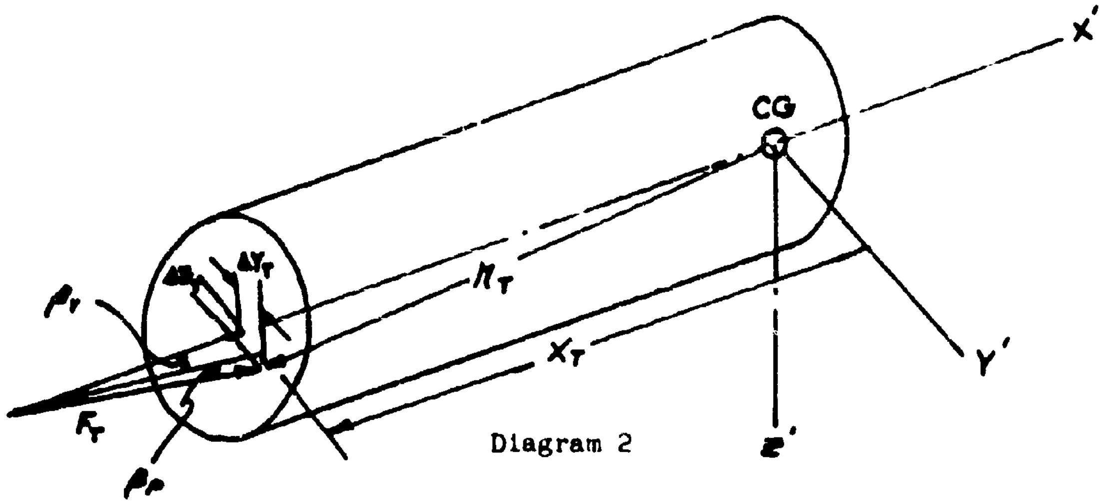

Diagram 2

The moment is expressed as:

$$\overline{M}_{Th} = \vec{r}_T \times \vec{F}_T$$

where from diagram 2:

$$\vec{r}_T = -\vec{i}' X_T + \vec{j}' \Delta Y_T + \vec{k}' \Delta Z_T$$

$$\vec{F}_T = \vec{i}' F_T \cos\beta_P \cos\beta_Y + \vec{j}' F_T \cos\beta_P \sin\beta_Y + \vec{k}' F_T \sin\beta_P$$

Taking the cross product, the moment becomes:

$$(15) \quad \vec{M}_{th} = \vec{i}'\left[(\vec{F}_T\cdot\vec{k}')\Delta Y_T - (\vec{F}_T\cdot\vec{j}')\Delta Z_T\right] + \vec{j}'\left[(\vec{F}_T\cdot\vec{i}')\Delta Z_T + (\vec{F}_T\cdot\vec{k}')X_T\right] - \vec{k}'\left[(\vec{F}_T\cdot\vec{j}')X_T + (\vec{F}_T\cdot\vec{i}')\Delta Y_T\right]$$

The body axes components of the moment due to the thrust vector are then:

$$M_{T_{roll}} = \Delta Y_T F_T \sin\beta_P - \Delta Z_T F_T \cos\beta_P \sin\beta_Y$$

$$(16) \quad M_{T_{pitch}} = \Delta Z_T F_T \cos\beta_P \cos\beta_Y + X_T F_T \sin\beta_P$$

$$M_{T_{yaw}} = -X_T F_T \cos\beta_P \sin\beta_Y - \Delta Y_T F_T \cos\beta_P \cos\beta_Y$$

4. <u>Damping Moment</u>

It remains only to define the aerodynamic damping moments about the X', Y' and Z' axes and these are $Md_{X'}$, $Md_{Y'}$, and $Md_{Z'}$ respectively.

The total applied moment is the sum of the individual moments derived previously, for equation 6.

Therefore:

<u>Body roll moment</u>

$$\sum M_{roll} = \frac{C_N Q A D}{|\sin\alpha|}\left\{\frac{\Delta CP_Y}{D}\left[-\frac{\dot{X}}{|V|}(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi) - \frac{\dot{Y}}{|V|}(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi) - \frac{\dot{Z}}{|V|}\cos\phi\cos\theta\right]\right. \tag{17a}$$

$$\left. + \frac{\Delta CP_Z}{D}\left[\frac{\dot{X}}{|V|}(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi) + \frac{\dot{Y}}{|V|}(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi) + \frac{\dot{Z}}{|V|}\sin\phi\cos\theta\right]\right\}$$

$$+ h\left[-F_{I} + F_{II} + F_{III} - F_{IV}\right]$$

$$+ \Delta Y_T F_T \sin\beta_P - \Delta Z_T F_T \cos\beta_P \sin\beta_Y - Md_{X'}$$

<u>Body pitch moment</u>

$$\sum M_{pitch} = -D_0 \Delta CP_{B} + X_c\left[F_{I} + F_{II}\right]$$

$$+ \frac{C_N Q A D}{|\sin\alpha|}\left(\frac{CG - CP}{D}\right)\left[-\frac{\dot{X}}{|V|}(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi) - \frac{\dot{Y}}{|V|}(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi) - \frac{\dot{Z}}{|V|}\cos\phi\cos\theta\right] \tag{17b}$$

$$+ \Delta Z_T F_T \cos\beta_P \cos\beta_Y + X_T F_T \sin\beta_P - Md_{Y'}$$

Body Yaw Moment

$$\sum M_{YAW} = D_0\,\Delta CP_Y - X_C\left[F_I + F_{III}\right]$$
$$+ \frac{C_N Q A D}{|\sin\alpha|}\left(\frac{CG-CP}{D}\right)\left[\frac{\dot{X}}{|V|}(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi)\right.$$
$$(17c) \qquad + \frac{\dot{Y}}{|V|}(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi)$$
$$\left. + \frac{\dot{Z}}{|V|}\sin\phi\cos\theta\right]$$
$$- X_T F_T \cos\beta_P \sin\beta_Y - \Delta Y_T F_T \cos\beta_P \cos\beta_Y - M_{dZ'}$$

5. **Total Rotational Equations of Motion**

The total rotational equation about the roll axis is found by equating the ($\sum M_{ROLL}$ term) in equation 17a to that in equation 5a and dividing by roll moment of inertia.

$$\ddot{\phi} - \ddot{\psi}\sin\theta - \dot{\psi}\dot{\theta}\cos\theta$$
$$+ \frac{I_Z - I_Y}{I_{X'}}\left\{\dot{\theta}\dot{\psi}(\cos^2\phi - \sin\phi)\cos\theta + \dot{\psi}^2\cos^2\theta\sin\phi\cos\phi - \dot{\theta}^2\cos\phi\sin\phi\right\}$$
$$- \frac{J_{X'Z'}}{I_{X'}}(\ddot{\psi}\cos\theta\cos\phi - 2\dot{\psi}\dot{\theta}\sin\theta\cos\phi - \ddot{\theta}\sin\phi - \dot{\psi}^2\cos\theta\sin\theta\sin\phi)$$
$$+ \frac{J_{X'Y'}}{I_{X'}}(2\dot{\psi}\dot{\theta}\sin\theta\sin\phi - \dot{\psi}^2\cos\theta\sin\theta\cos\phi - \ddot{\theta}\cos\phi - \ddot{\psi}\cos\theta\sin\phi)$$
$$(18a) \quad + \frac{J_{Y'Z'}}{I_{X'}}\left\{\dot{\psi}^2\cos^2\theta(\cos^2\phi - \sin^2\phi) - 4\dot{\theta}\dot{\psi}\cos\phi\cos\theta\sin\phi + \dot{\theta}^2(\sin^2\phi - \cos^2\phi)\right\}$$
$$+ \frac{C_N Q A D}{I_{X'}|\sin\alpha|}\left\{\frac{\Delta CP_X}{D}\left[\frac{\dot{X}}{|V|}(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)\right.\right.$$
$$+ \frac{\dot{Y}}{|V|}(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)$$
$$\left. + \frac{\dot{Z}}{|V|}\cos\phi\cos\theta\right]$$

$$-\frac{\Delta CP_B}{D}\left[\frac{\dot{x}}{|V|}(\sin\phi\sin\theta\cos\psi-\cos\phi\sin\psi)+\frac{\dot{y}}{|V|}(\cos\phi\cos\psi+\sin\phi\sin\theta\sin\psi)+\frac{\dot{z}}{|V|}\sin\phi\cos\theta\right]\Bigg\}\quad-\frac{h}{I_{x'}}\left[-F_{I}+F_{II}+F_{III}-F_{IV}\right]$$

$$-\frac{\Delta Y_T F_T}{I_{x'}}\sin\beta_P+\frac{\Delta Z_T F_T}{I_{x'}}\cos\beta_P\sin\beta_Y+\frac{Md_{x'}}{I_{x'}}=0$$

Likewise equating the (ΣM_{pitch} term) in equation 17b to that in equation 5b and dividing by the pitch axis moment of inertia, produces the body pitch axis rotational equation of motion.

$$\ddot{\theta}\cos\phi-\dot{\theta}\dot{\phi}\sin\phi+\ddot{\psi}\cos\theta\sin\phi-\dot{\psi}\dot{\theta}\sin\theta\sin\phi+\dot{\psi}\dot{\phi}\cos\theta\cos\phi$$

$$+\left(\frac{I_{x'}-I_{z'}}{I_{y'}}\right)\left(\dot{\phi}\dot{\psi}\cos\theta\cos\phi-\dot{\phi}\dot{\theta}\sin\phi-\dot{\psi}^2\sin\theta\cos\theta\cos\phi+\dot{\psi}\dot{\theta}\sin\theta\sin\phi\right)$$

$$+\frac{J_{x'z'}}{I_{y'}}\left(\dot{\phi}^2-2\dot{\phi}\dot{\psi}\sin\theta+\dot{\psi}^2\sin^2\theta-\dot{\psi}^2\cos^2\theta\cos^2\phi+2\dot{\psi}\dot{\theta}\cos\theta\cos\phi\sin\phi-\dot{\theta}^2\sin^2\phi\right)$$

$$-\frac{J_{x'y'}}{I_{y'}}\left(\ddot{\phi}-\ddot{\psi}\sin\theta-2\dot{\theta}\dot{\psi}\cos\theta\sin^2\phi-\dot{\theta}^2\sin\phi\cos\phi+\dot{\psi}^2\cos^2\theta\sin\phi\cos\phi\right)$$

(18b)

$$+\frac{J_{y'z'}}{I_{y'}}\left(\ddot{\theta}\sin\phi+2\dot{\theta}\dot{\phi}\cos\phi+2\dot{\phi}\dot{\psi}\cos\theta\sin\phi-\ddot{\psi}\cos\theta\cos\phi-\dot{\psi}^2\sin\theta\cos\theta\sin\phi\right)$$

$$+\frac{D_0\,\Delta CP_B}{I_{y'}}-\frac{x_c}{I_{y'}}\left[F_{II}+F_{III}\right]+\frac{Md_{y'}}{I_{y'}}$$

$$+\frac{C_N QAD}{I_{y'}\sin\alpha}\left(\frac{CG-CP}{D}\right)\left[\frac{\dot{x}}{|V|}(\cos\phi\sin\theta\cos\psi+\sin\phi\sin\psi)+\frac{\dot{y}}{|V|}(\cos\phi\sin\theta\sin\psi-\sin\phi\cos\psi)+\frac{\dot{z}}{|V|}\cos\phi\cos\theta\right]$$

$$-\frac{\Delta Z_T F_T}{I_{y'}}\cos\beta_P\cos\beta_Y-\frac{X_T F_T}{I_{y'}}\sin\beta_P=0$$

The rotational equation about the body yaw axis is derived by equating the (ΣM_{yaw} term) in equation 17c to that in equation 5c and dividing by the yaw moment of inertia.

$$\ddot{\psi}\cos\theta\cos\phi - \dot{\psi}\dot{\theta}\sin\theta\cos\phi - \dot{\phi}\dot{\psi}\cos\theta\sin\phi - \ddot{\theta}\sin\phi - \dot{\theta}\dot{\phi}\cos\phi$$

$$+\left(\frac{I_{Y'}-I_{X'}}{I_{Z'}}\right)\left(\dot{\phi}\dot{\theta}\cos\phi + \dot{\phi}\dot{\psi}\cos\theta\sin\phi - \dot{\psi}\dot{\theta}\sin\theta\cos\phi - \dot{\psi}^2\cos\theta\sin\theta\sin\phi\right)$$

$$-\frac{J_{X'Z'}}{I_{Z'}}\left(\ddot{\phi} - \ddot{\psi}\sin\theta + \dot{\theta}^2\cos\phi\sin\phi - 2\dot{\theta}\dot{\psi}\cos\theta\cos^2\phi - \dot{\psi}^2\cos^2\theta\sin\phi\cos\phi\right)$$

$$-\frac{J_{Y'Z'}}{I_{Z'}}\left(\ddot{\theta}\cos\phi + 2\dot{\phi}\dot{\psi}\cos\theta\cos\phi - 2\dot{\theta}\dot{\phi}\sin\phi - \dot{\psi}^2\sin\theta\cos\theta\cos\phi + \ddot{\psi}\cos\theta\sin\phi\right)$$

$$+\frac{J_{X'Y'}}{I_{Z'}}\left(\dot{\theta}^2\cos^2\phi + 2\dot{\theta}\dot{\psi}\cos\phi\cos\theta\sin\phi - \dot{\phi}^2 + 2\dot{\phi}\dot{\psi}\sin\theta + \dot{\psi}^2\left\{\cos^2\theta\sin\phi - \sin^2\theta\right\}\right)$$

(18c)

$$-\frac{D_o\Delta CP_Y}{I_{Z'}} + \frac{X_c}{I_{Z'}}\left[F_I + F_{II}\right] + \frac{m d_{Z'}}{I_{Z'}}$$

$$-\frac{C_N QAD}{|\sin\alpha| I_{Z'}}\left(\frac{CG-CP}{D}\right)\left[\frac{\dot{X}}{|V|}\left(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi\right) + \frac{\dot{Y}}{|V|}\left(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi\right) + \frac{\dot{Z}}{|V|}\sin\phi\cos\theta\right]$$

$$+\frac{X_T F_T\cos\beta_P\sin\beta_Y}{I_{Z'}} + \frac{\Delta Y_T F_T}{I_{Z'}}\cos\beta_P\cos\beta_Y = 0$$

Section II - Translational Equations of Motion

The total translational equations of motion, written with respect to the space fixed reference system are now derived. The total acceleration along any space fixed axis will be due to the sum of the aerodynamic force, the weight component, the control forces, and the thrust component along that particular axis, divided by the mass. The force vector equations are:

$$
(19)\quad
\begin{aligned}
F_x &= \vec{F}_{AERO}\cdot\vec{i} + \vec{Mg}\cdot\vec{i} + \vec{F}_c\cdot\vec{i} + \vec{T}\cdot\vec{i}\\
F_Y &= \vec{F}_{AERO}\cdot\vec{j} + \vec{Mg}\cdot\vec{j} + \vec{F}_c\cdot\vec{j} + \vec{T}\cdot\vec{j}\\
F_z &= \vec{F}_{AERO}\cdot\vec{K} + \vec{Mg}\cdot\vec{K} + \vec{F}_c\cdot\vec{K} + \vec{T}\cdot\vec{K}
\end{aligned}
$$

Aerodynamic Forces

By means of a vector transformation the body aerodynamic forces are defined along space fixed axes as:

$$
(20)\quad
\begin{aligned}
F_{x_{aero}} &= \vec{F}_{aero}\cdot\vec{i} = -D_0\cos\theta\cos\psi + (\vec{F}_N\cdot\vec{j}')(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi)\\
&\quad + (\vec{F}_N\cdot\vec{K}')(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)\\
F_{y_{aero}} &= \vec{F}_{aero}\cdot\vec{j} = -D_0\cos\theta\sin\psi + (\vec{F}_N\cdot\vec{j}')(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi)\\
&\quad + (\vec{F}_N\cdot\vec{K}')(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)\\
F_{z_{aero}} &= \vec{F}_{aero}\cdot\vec{k} = D_0\sin\theta + (\vec{F}_N\cdot\vec{j}')\sin\phi\cos\theta\\
&\quad + (\vec{F}_N\cdot\vec{K}')\cos\phi\cos\theta
\end{aligned}
$$

Substituting equations 8 into equations 20 results in the following:

$$
(21)\ a\quad
\begin{aligned}
F_{x_{AERO}} &= -D_0\cos\theta\cos\psi\\
&\quad - \frac{C_N Q A}{|\sin\alpha|}\Big[\frac{\dot{x}}{|V|}(\sin^2\psi + \sin^2\theta\cos^2\psi)\\
&\quad + \frac{\dot{Y}}{|V|}(\sin^2\theta\sin\psi\cos\psi - \cos\psi\sin\psi)\\
&\quad + \frac{\dot{z}}{|V|}\cos\theta\sin\theta\cos\psi\Big]
\end{aligned}
$$

(21b)

$$F_{Y_{AERO.}} = -D_0 \cos\theta \sin\psi - \frac{C_N Q A}{|\sin\alpha|}\left[\frac{\dot{X}}{|V|}\left(\sin^2\theta \sin\psi \cos\psi - \sin\psi \cos\psi\right) + \frac{\dot{Y}}{|V|}\left(\cos^2\psi + \sin^2\theta \sin^2\psi\right) + \frac{\dot{Z}}{|V|}\cos\theta \sin\theta \sin\psi\right]$$

(21c)

$$F_{Z_{AERO}} = D_0 \sin\theta - \frac{C_N Q A}{|\sin\alpha|}\left[\frac{\dot{X}}{|V|}\sin\theta \cos\psi \cos\theta + \frac{\dot{Y}}{|V|}\sin\theta \sin\psi \cos\theta + \frac{\dot{Z}}{|V|}\cos^2\theta\right]$$

Weight

The vehicle weight, space fixed components are: (ignoring trajectory twist)

(22)

$$Mg_X = -Mg \sin\Omega$$
$$Mg_Y = 0$$
$$Mg_Z = Mg \cos\Omega$$

Control Force

The control force components along space fixed axes are:

(23)

$$F_{cX} = (F_I + F_{III})(\sin\phi \sin\theta \cos\psi - \cos\phi \sin\psi) + (F_{II} + F_{IV})(\cos\phi \sin\theta \cos\psi + \sin\phi \sin\psi)$$

$$F_{cY} = (F_I + F_{IV})(\cos\phi \cos\psi + \sin\phi \sin\theta \sin\psi)\;(F_{II} + F_{IV})(\cos\phi \sin\theta \sin\psi - \sin\phi \cos\psi)$$

$$F_{cZ} = (F_I + F_{III})\sin\phi \cos\theta + (F_{II} + F_{IV})\cos\phi \cos\theta$$

Thrust

The thrust vector space fixed axes components are:

(24)

$$F_{T_X} = F_T \cos\beta_P \cos\beta_Y \cos\theta \cos\psi + F_T \cos\beta_P \sin\beta_Y (\sin\phi \sin\theta \cos\psi - \cos\phi \sin\psi) + F_T \sin\beta_P (\cos\phi \sin\theta \cos\psi + \sin\phi \sin\psi)$$

$$F_{T_Y} = F_T \cos\beta_P \cos\beta_Y \cos\theta \sin\psi \quad F_T \cos\beta_P \sin\beta_Y (\cos\phi \cos\psi + \sin\phi \sin\theta \sin\psi) \quad F_T \sin\beta_P (\cos\phi \sin\theta \sin\psi - \sin\phi \cos\psi)$$

$$F_{T_Z} = -F_T \cos\beta_P \cos\beta_Y \sin\theta + F_T \cos\beta_P \sin\beta_Y \sin\phi \cos\theta + F_T \sin\beta_P \cos\phi \cos\theta$$

The Total Translational Equations

The space fixed accelerations are now presented as the sum of components of equations 21, 22, 23, and 24 divided by the mass term.

(25a)

$$\ddot{X} = -\frac{D_0}{m} \cos\theta \cos\psi - g \sin\Omega - \frac{C_N Q A}{m \sin\alpha} \left[\frac{\dot{x}}{|V|} (\sin^2\psi + \sin^2\theta \cos^2\psi) + \frac{\dot{y}}{|V|} (\sin^2\theta \sin\psi \cos\psi - \cos\psi \sin\psi) + \frac{\dot{z}}{|V|} \cos\theta \sin\theta \cos\psi \right]$$

$$+ \frac{(F_{II} + F_{III})}{m} (\sin\phi \sin\theta \cos\psi - \cos\phi \sin\psi)$$

$$\frac{(F_{II} + F_{III})}{m} (\cos\phi \sin\theta \cos\psi + \sin\phi \sin\psi)$$

$$+ \frac{F_I}{m} (\cos\beta_P \cos\beta_Y) \cos\theta \cos\psi$$

$$+ \frac{F_I}{m} \cos\beta_P \sin\beta_Y (\sin\phi \sin\theta \cos\psi - \cos\phi \sin\psi)$$

$$+ \frac{F_I}{m} \sin\beta_P (\cos\phi \sin\theta \cos\psi + \sin\phi \sin\psi)$$

$$
(25b)\quad \ddot{Y} = -\frac{D}{m}\cos\theta\sin\psi - \frac{C_N QA}{m|\sin\alpha|}\Big[\frac{\dot{X}}{|V|}(\sin^2\theta\sin\psi\cos\psi - \sin\psi\cos\psi) + \frac{\dot{Y}}{|V|}(\cos^2\psi + \sin^2\theta\sin^2\psi) + \frac{\dot{Z}}{|V|}\cos\theta\sin\theta\sin\psi\Big]
$$

$$
+ \frac{F_I + F_{III}}{m}(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi) + \frac{F_{II} + F_{IV}}{m}(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)
$$

$$
+ \frac{F_T}{m}\cos\beta_z\cos\beta_y\cos\theta\sin\psi
$$

$$
+ \frac{F_T}{m}\cos\beta_z\sin\beta_y(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi)
$$

$$
+ \frac{F_T}{m}\sin\beta_z(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)
$$

$$
(25c)\quad \ddot{Z} = \frac{D}{m}\sin\theta + g\cos\Omega
$$

$$
- \frac{C_N QA}{m|\sin\alpha|}\Big[\frac{\dot{X}}{|V|}\sin\theta\cos\psi\cos\theta + \frac{\dot{Y}}{|V|}\sin\theta\sin\psi\cos\theta + \frac{\dot{Z}}{|V|}\cos^2\theta\Big]
$$

$$
+ \frac{F_I + F_{III}}{m}\sin\phi\cos\theta + \frac{F_{II} + F_{IV}}{m}\cos\phi\cos\theta - \frac{F_T}{m}\cos\beta_z\cos\beta_y\sin\theta
$$

$$
+ \frac{F_T}{m}\cos\beta_z\sin\beta_y\sin\phi\cos\theta + \frac{F_T}{m}\sin\beta_z\cos\phi\cos\theta
$$

Section III - Approximate Forms for Equations of Motion.

Preliminary Simplification

Although equations 18 and 25 are total equations and may be programmed for very large digital and analog computer systems, they are complex and cumbersome and can be simplified. The first step in simplifying these equations would be to reduce the number of terms in equations 18 by assuming that products of inertia are negligible and that $I_{Y_1} \cong I_{Z_1} \cong I$ for most symmetrical ballistic missiles. Further simplification is achieved by expressing body axes moment along the $X_1\ Y_1\ Z_1$ axes with the following equations:

$$M_{x_1} = M_{x'}$$

(26) $$M_{Y_1} = M_{Y'} \cos\phi - M_{Z'} \sin\phi$$

$$M_{Z_1} = M_{Z'} \cos\phi + M_{Y'} \sin\phi$$

Incorporating the above assumption and substituting the expressions for $M_{x'}$, $M_{y'}$ and $M_{z'}$ into equations 26, yields the following rotational equations:

(27a)
$$\ddot{\phi} - \ddot{\psi}\sin\theta - \dot{\psi}\dot{\theta}\cos\theta$$
$$+ \frac{C_N Q A D}{I_x |\sin\alpha|} \Bigg\{ \frac{\Delta CP_Y}{D} \Bigg[\frac{\dot{x}}{|V|}(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi) + \frac{\dot{Y}}{|V|}(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi) + \frac{\dot{z}}{|V|}\cos\phi\cos\theta \Bigg]$$
$$- \frac{\Delta CP_Z}{D} \Bigg[\frac{\dot{x}}{|V|}(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi) + \frac{\dot{Y}}{|V|}(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi) + \frac{\dot{z}}{|V|}\sin\phi\cos\theta \Bigg] \Bigg\} - \frac{h}{I_{x'}}\left[-F_I + F_{II} + F_{III} - F_{IV}\right]$$
$$- \frac{\Delta Y_T F_T}{I_{x'}}\sin\beta_P + \frac{\Delta Z_T F_T}{I_{x'}}\cos\beta_P\sin\beta_Y + \frac{M_{dx'}}{I_{x'}} = 0$$

(27b)

$$\ddot{\theta} + \dot{\psi}^2 \cos\theta \sin\theta + \frac{I_{x'}}{I}\left[\dot{\phi}\dot{\psi}\cos\theta - \dot{\psi}^2 \cos\theta \sin\theta\right]$$

$$+ D_0\left(\frac{\Delta CP_Z}{I}\cos\phi + \frac{\Delta CP_Y}{I}\sin\phi\right)$$

$$+ \frac{C_N QAD}{I|\sin\alpha|}\left(\frac{CG-CP}{D}\right)\left[\frac{\dot{X}}{|V|}\sin\theta\cos\psi + \frac{\dot{Y}}{|V|}\sin\theta\sin\psi + \frac{\dot{Z}}{|V|}\cos\theta\right]$$

$$-\frac{X_C}{I}\left[\cos\phi\left(F_{II} + F_{III}\right) + \sin\phi\left(F_I + F_{III}\right)\right]$$

$$-\frac{X_T}{I}F_T\left(\sin\beta_P\cos\phi + \cos\beta_P\sin\beta_Y\sin\phi\right)$$

$$-\frac{\Delta Z_T}{I}\left[F_T\cos\beta_P\cos\beta_Y\cos\phi\right] - \frac{\Delta Y_T}{I}\left[F_T\cos\beta_P\cos\beta_Y\sin\phi\right]$$

$$+\frac{M d_{Y'}}{I}\cos\phi - \frac{M d_{Z'}}{I}\sin\phi = 0$$

(27c)

$$\ddot{\psi}\cos\theta - 2\dot{\psi}\dot{\theta}\sin\theta + \frac{I_{x'}}{I}\left[-\dot{\phi}\dot{\theta} + \dot{\psi}\dot{\theta}\sin\theta\right]$$

$$+\frac{D_0}{I}\left[\Delta CP_Z\sin\phi - \Delta CP_Y\cos\phi\right] + \frac{\Delta Y_T}{I}F_T\cos\beta_P\cos\beta_Y\cos\phi$$

$$+\frac{C_N QAD}{I|\sin\alpha|}\left(\frac{CG-CP}{D}\right)\left[\frac{\dot{X}}{|V|}\sin\psi - \frac{\dot{Y}}{|V|}\cos\psi\right]$$

$$+\frac{X_C}{I}\left[\cos\phi\left(F_I + F_{III}\right) - \sin\phi\left(F_{II} + F_{III}\right)\right] - \frac{\Delta Z_T}{I}F_T\cos\beta_P\cos\beta_Y\sin\phi$$

$$+\frac{X_T}{I}\left[F_T\cos\beta_P\sin\beta_Y\cos\phi - F_T\sin\beta_P\sin\phi\right]$$

$$+\frac{M d_{Z'}\cos\phi}{I} + \frac{M d_{Y'}}{I}\sin\phi = 0$$

Perturbation Equations

The perturbation equations are obtained by subtracting a standard, zero angle of attack trajectory, from equations (25) and (27). The standard trajectory can be described by the following parameters:

$$\alpha = \beta = \psi = \dot{\psi} = \ddot{\psi} = \dot{\alpha} = \dot{\beta} = Y = \dot{Y} = \ddot{Y} = \phi = \dot{\phi} = \ddot{\phi} = 0$$

Pitch angle, $\theta_s = \frac{\pi}{2} - \bar{\Phi}_t$, and for all but the steepest trajectories: $\dot{\theta}_s = -\dot{\bar{\Phi}}_t \cong 0 \qquad \ddot{\theta}_s = -\ddot{\bar{\Phi}}_t \cong 0$

and therefore:

$$\sin\theta_s = \cos\bar{\Phi}_t \qquad z = z_s \qquad X = X_s$$
$$\cos\theta_s = \sin\bar{\Phi}_t \qquad \dot{z} = \dot{z}_s \qquad \dot{X} = \dot{X}_s$$
$$\ddot{z} = \ddot{z}_s \qquad \ddot{X} = \ddot{X}_s$$

Control forces and misalignments are zero also.

Substitution of these parameters into equations 25 and 27 gives the standard trajectory equations as:

$$(28) \quad \begin{aligned} &\ddot{X}_s + \frac{D_e}{m}\cos\theta_s + g\sin\Omega_s - \frac{F_T}{m}\cos\theta_s = 0 \\ &\ddot{Y}_s = 0 \\ &\ddot{z}_s - \frac{D_e}{m}\sin\theta_s - g\cos\Omega_s + \frac{F_T}{m}\sin\theta_s = 0 \\ &\ddot{\phi}_s = \ddot{\psi}_s = \ddot{\theta}_s = 0 \end{aligned}$$

Subtracting equations (28) from equations 25 and 27 give the perturbation equations of motion as: (assuming $\Omega_s \cong \Omega$)

$$(29a) \quad \begin{aligned} &\Delta\ddot{X} + \frac{D_e}{m}\left(\cos\theta\cos\psi - \cos\theta_s\right) \\ &+ \frac{C_N Q A}{m \sin\alpha}\left[\frac{\dot{X}}{|V|}\left(\sin^2\psi + \sin^2\theta\cos^2\psi\right) + \frac{\dot{Y}}{|V|}\left(\sin^2\theta\sin\psi\cos\psi - \cos\psi\sin\psi\right) \right. \\ &\qquad \left. + \frac{\dot{z}}{|V|}\cos\theta\sin\theta\cos\psi\right] \\ &- \left(\frac{F_z + F_{zc}}{m}\right)\left(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi\right) \\ &- \left(\frac{F_{zc} + F_{zc}}{m}\right)\left(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi\right) \\ &- \frac{F_T}{m}\cos\beta_p\sin_y\gamma\left(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi\right) \\ &- \frac{F_T}{m}\sin\beta_p\left(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi\right) \\ &- \frac{F_T}{m}\left(\cos\beta_p\cos\beta_Y\cos\psi\cos\theta - \cos\theta_s\right) = 0 \end{aligned}$$

(29b)

$$\Delta\ddot{Y} + \frac{D_e}{m}\cos\theta\sin\psi$$

$$+ \frac{C_N Q A}{m|\sin\alpha|}\left[\frac{\dot{X}}{|V|}\left(\sin^2\theta\sin\psi\cos\psi - \sin\psi\cos\psi\right) + \frac{\dot{Y}}{|V|}\left(\cos^2\psi + \sin^2\theta\sin^2\psi\right) + \frac{\dot{Z}}{|V|}\cos\theta\sin\theta\sin\psi\right]$$

$$-\left(\frac{F_{I} + F_{III}}{m}\right)\left(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi\right)$$

$$-\left(\frac{F_{II} + F_{IV}}{m}\right)\left(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi\right) - \frac{F_T}{m}\cos\beta_r\cos\beta_p\cos\theta\sin\psi$$

$$-\frac{F_T}{m}\cos\beta_p\sin\beta_r\left(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi\right)$$

$$-\frac{F_T}{m}\sin\beta_p\left(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi\right) = 0$$

(29c)

$$\Delta\ddot{Z} - \frac{D_e}{m}\left(\sin\theta - \sin\theta_s\right)$$

$$+ \frac{C_N Q A}{m|\sin\alpha|}\left[\frac{\dot{X}}{|V|}\sin\theta\cos\psi\cos\theta + \frac{\dot{Y}}{|V|}\sin\theta\sin\psi\cos\theta + \frac{\dot{Z}}{|V|}\cos^2\theta\right]$$

$$-\left(\frac{F_{I} + F_{III}}{m}\right)\sin\phi\cos\theta - \left(\frac{F_{II} + F_{IV}}{m}\right)\cos\phi\cos\theta$$

$$+ \frac{F_T}{m}\left(\cos\beta_p\cos\beta_r\sin\theta - \sin\theta_s\right)$$

$$- \frac{F_T}{m}\left(\cos\beta_p\sin\beta_r\sin\phi\cos\theta + \sin\beta_p\cos\phi\cos\theta\right) = 0$$

(29d)

$$\ddot{\phi} - \ddot{\psi}\sin\theta - \dot{\psi}\dot{\theta}\cos\theta$$

$$+ \frac{C_N Q A D}{I_x|\sin\alpha|}\left\{\frac{\Delta CP_r}{D}\left[\frac{\dot{X}}{|V|}\left(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi\right) + \frac{\dot{Y}}{|V|}\left(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi\right) + \frac{\dot{Z}}{|V|}\cos\phi\cos\theta\right]\right.$$

$$
\begin{aligned}
&- \frac{\Delta C P_Z}{D}\Bigg[\frac{\dot{X}}{|V|}(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi) \\
&\qquad + \frac{\dot{Y}}{|V|}(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi) \\
&\qquad + \frac{\dot{Z}}{|V|}\sin\phi\cos\theta\Bigg]\Bigg\} \\
&- \frac{h}{I_x'}\left[-F_I + F_{II} + F_{III} - F_{IV}\right] \\
&- \frac{\Delta Y_T F_T}{I_x'}\sin\beta_P + \frac{\Delta Z_T F_T}{I_x'}\cos\beta_P\sin\beta_Y + \frac{M d_{x'}}{I_x'} = 0
\end{aligned}
$$

(29e)

$$
\begin{aligned}
&\ddot{\theta} + \dot{\psi}^2\cos\theta\sin\theta + \frac{I_x}{I}\left[\dot{\phi}\dot{\psi}\cos\theta - \dot{\psi}^2\cos\theta\sin\theta\right] \\
&+ D_o\left(\frac{\Delta C P_Z}{I}\cos\phi + \frac{\Delta C P_Y}{I}\sin\phi\right) \\
&+ \frac{C_N Q A D}{I|\sin\alpha|}\left(\frac{CG - CP}{D}\right)\left[\frac{\dot{X}}{|V|}\sin\theta\cos\psi + \frac{\dot{Y}}{|V|}\sin\theta\sin\psi + \frac{\dot{Z}}{|V|}\cos\theta\right] \\
&- \frac{X_c}{I}\left[\cos\phi\,(F_{II} + F_{IV}) + \sin\phi\,(F_I + F_{III})\right] \\
&- \frac{X_T}{I}F_T(\sin\beta_P\cos\phi + \cos\beta_P\sin\beta_Y\sin\phi) \\
&- \frac{\Delta Z_T}{I}F_T\cos\beta_P\cos\beta_Y\cos\phi - \frac{\Delta Y_T}{I}F_T\cos\beta_P\cos\beta_Y\sin\phi \\
&+ \frac{M d_{Y'}}{I}\cos\phi - \frac{M d_{Z'}}{I}\sin\phi = 0
\end{aligned}
$$

(29f)

$$\ddot{\psi}\cos\theta - 2\dot{\psi}\dot{\theta}\sin\theta + \frac{I_K}{I}\left[-\dot{\phi}\dot{\theta} + \dot{\psi}\dot{\theta}\sin\theta\right]$$

$$+ \frac{D_o}{I}\left[\Delta CP_Z \sin\phi - \Delta CP_Y \cos\phi\right] + \frac{\Delta Y_T}{I} F_T \cos\beta_P \cos\beta_Y \cos\phi$$

$$+ \frac{C_N QAD}{I|\sin\alpha|}\left(\frac{CG-CP}{D}\right)\left[\frac{\dot{X}}{|V|}\sin\psi - \frac{\dot{Y}}{|V|}\cos\psi\right]$$

$$+ \frac{X_c}{I}\left[\cos\phi\,(F_I + F_{III}) - \sin\phi\,(F_{II} + F_{IV})\right]$$

$$- \frac{\Delta Z_T}{I} F_T \cos\beta_P \cos\beta_Y \sin\phi$$

$$+ \frac{X_T}{I}\left(F_T \cos\beta_P \sin\beta_Y \cos\phi - F_T \sin\beta_P \sin\phi\right)$$

$$+ \frac{m d_Z'}{I}\cos\phi + \frac{m d_Y'}{I}\sin\phi = 0$$

Linearized Perturbation Equations

Further simplification is achieved when all angles except θ, ϕ, Ω and θ_S can be linearized.

Incorporating this assumption with equations 29, the linearized perturbation equations of motion are obtained as:

(30a)

$$\Delta\ddot{X} + \frac{D_o}{m}\left(\cos\theta - \cos\theta_S\right)$$

$$+ \frac{C_{N\alpha} QA}{m}\left[\frac{\dot{X}}{|V|}\sin^2\theta + \frac{\dot{Y}}{|V|}\left(\psi \sin^2\theta - \psi\right) + \frac{\dot{Z}}{|V|}\cos\theta\sin\theta\right]$$

$$
\begin{aligned}
&-\left(\frac{F_I+F_{III}}{m}\right)\left(\sin\phi\sin\theta-\psi\cos\phi\right)-\left(\frac{F_{II}+F_{IV}}{m}\right)\left(\cos\phi\sin\theta+\psi\sin\phi\right)\\
&-\frac{F_T}{m}\left(\cos\theta-\cos\theta_s\right)-\frac{F_T}{m}\beta_Y\left(\sin\phi\sin\theta-\psi\cos\phi\right)\\
&-\frac{F_T}{m}\beta_P\left(\cos\phi\sin\theta+\psi\sin\phi\right)=0
\end{aligned}
$$

(30b)

$$
\begin{aligned}
&\Delta\ddot{Y}+\frac{D_0}{m}\psi\cos\theta\\
&\quad+\frac{C_{N_\alpha}QA}{m}\left[\frac{\dot{X}}{|V|}\left(\psi\sin^2\theta-\psi\right)+\frac{\dot{Y}}{|V|}+\frac{\dot{Z}}{|V|}\cos\theta\sin\theta\,\psi\right]\\
&\quad-\left(\frac{F_I+F_{III}}{m}\right)\left(\cos\phi+\sin\phi\sin\theta\cdot\psi\right)-\frac{F_T}{m}\psi\cos\theta\\
&\quad-\left(\frac{F_{II}+F_{IV}}{m}\right)\left(\cos\phi\sin\theta\cdot\psi-\sin\phi\right)-\frac{F_T}{m}\beta_Y\left(\cos\phi+\psi\sin\phi\sin\theta\right)\\
&\quad-\frac{F_T}{m}\beta_P\left(\cos\phi\sin\theta\cdot\psi-\sin\phi\right)=0
\end{aligned}
$$

(30c)

$$
\begin{aligned}
&\Delta\ddot{Z}-\frac{D_0}{m}\left[\sin\theta-\sin\theta_s\right]\\
&\quad+\frac{C_{N_\alpha}QA}{m}\left[\frac{\dot{X}}{|V|}\sin\theta\cos\theta+\frac{\dot{Y}}{|V|}\psi\sin\theta\cos\theta+\frac{\dot{Z}}{|V|}\cos^2\theta\right]\\
&\quad-\left(\frac{F_I+F_{III}}{m}\right)\sin\phi\cos\theta-\left(\frac{F_{II}+F_{IV}}{m}\right)\cos\phi\cos\theta\\
&\quad+\frac{F_T}{m}\left(\sin\theta-\sin\theta_s\right)-\frac{F_T}{m}\beta_Y\sin\phi\cos\theta\\
&\quad-\frac{F_T}{m}\beta_P\cos\phi\cos\theta=0
\end{aligned}
$$

(30d)

$$
\begin{aligned}
&\ddot{\phi} - \ddot{\psi}\sin\theta - \dot{\theta}\dot{\psi}\cos\theta \\
&+ \frac{C_{N_\alpha} QAD}{I_{x'}} \Bigg\{ \frac{\Delta CP_Y}{D} \Bigg[\frac{\dot{X}}{|V|}(\cos\phi\sin\theta + \psi\sin\phi) \\
&\qquad + \frac{\dot{Y}}{|V|}(\cos\phi\sin\theta\cdot\psi - \sin\phi) \\
&\qquad + \frac{\dot{Z}}{|V|}\cos\phi\cos\theta \Bigg] \\
&\quad - \frac{\Delta CP_Z}{D} \Bigg[\frac{\dot{X}}{|V|}(\sin\phi\sin\theta - \psi\cos\phi) \\
&\qquad + \frac{\dot{Y}}{|V|}(\cos\phi + \sin\phi\sin\theta\cdot\psi) \\
&\qquad + \frac{\dot{Z}}{|V|}\sin\phi\cos\theta \Bigg] \Bigg\} \\
&- \frac{h}{I_{x'}}\left[-F_{I} + F_{II} + F_{III} - F_{IV}\right] - \frac{\Delta Y_T}{I_{x'}} F_T \beta_P \\
&+ \frac{\Delta Z_T}{I_{x'}} F_T \beta_Y + \frac{M_{dx'}}{I_{x'}} = 0
\end{aligned}
$$

(30e)

$$
\begin{aligned}
&\ddot{\theta} + \frac{I_{x'}}{I}\left[\dot{\phi}\dot{\psi}\cos\theta - \dot{\psi}^2\cos\theta\sin\theta\right] \\
&+ \dot{\psi}^2\cos\theta\sin\theta + D_0\left(\frac{\Delta CP_Z}{I}\cos\phi + \frac{\Delta CP_Y}{I}\sin\phi\right) \\
&+ \frac{C_{N_\alpha} QAD}{I}\left(\frac{CG-CP}{D}\right)\left[\frac{\dot{X}}{|V|}\sin\theta + \frac{\dot{Y}}{|V|}\psi\sin\theta + \frac{\dot{Z}}{|V|}\cos\theta\right] \\
&- \frac{X_C}{I}\left[\cos\phi\left(F_{II} + F_{IV}\right) + \sin\phi\left(F_{I} + F_{III}\right)\right] \\
&- \frac{X_T}{I} F_T\left(\beta_P\cos\phi + \beta_Y\sin\phi\right) - \frac{\Delta Z_T}{I} F_T\cos\phi \\
&- \frac{\Delta Y_T}{I} F_T\sin\phi + \frac{M_{dy'}}{I}\cos\phi - \frac{M_{dz'}}{I}\sin\phi = 0
\end{aligned}
$$

$$\ddot{\psi}\cos\theta - 2\dot{\psi}\dot{\theta}\sin\theta + \frac{I_X}{I}\left[-\dot{\phi}\dot{\theta} + \dot{\psi}\dot{\theta}\sin\theta\right]$$

(30f) $$+D_0\left[\frac{\Delta CP_Z}{I}\sin\phi - \frac{\Delta CP_Y}{I}\cos\phi\right] + \frac{C_{N_\alpha}QAD}{I}\left(\frac{CG-CP}{D}\right)\left[\frac{\dot{X}}{|V|}\psi - \frac{\dot{Y}}{|V|}\right]$$

$$+ \frac{X_C}{I}\left[(F_I + F_{III})\cos\phi - (F_{II} + F_{IV})\sin\phi\right]$$

$$- \frac{\Delta Z_I}{I}F_T\sin\phi + \frac{\Delta Y_I}{I}F_T\cos\phi + \frac{X_I}{I}F_T\beta_Y\cos\phi$$

$$- \frac{X_I}{I}F_T\beta_P\sin\phi + \frac{M_{d_Z'}}{I}\cos\phi + \frac{M_{d_Y'}}{I}\sin\phi = 0$$

<u>Derivation of the components of angle of attack</u>

The normal force, F_N, can be written in space fixed components by expressing $|\vec{V}\times\vec{i}'|\times\vec{i}'$ along space fixed axes. The final expression is the following:

(31) $$\vec{F}_N = C_{N_\alpha}QA\Big[-\vec{i}\left(\frac{\dot{X}}{|V|}\sin^2\theta - \frac{\dot{Y}}{|V|}\psi\cos^2\theta + \frac{\dot{Z}}{|V|}\cos\theta\sin\theta\right)$$
$$+\vec{j}\left(\frac{\dot{X}}{|V|}\psi\cos^2\theta - \frac{\dot{Y}}{|V|} - \frac{\dot{Z}}{|V|}\psi\sin\theta\cos\theta\right)$$
$$-\vec{k}\left(\frac{\dot{X}}{|V|}\sin\theta\cos\theta + \frac{\dot{Y}}{|V|}\psi\sin\theta\cos\theta + \frac{\dot{Z}}{|V|}\cos^2\theta\right)\Big]$$

Because ψ is small, the component along the $\vec{k}$ axis is approximately equal to $C_{N_\alpha}QA\alpha_P\cos\theta$ and the j component is approximately equal to $C_{N_\alpha}QA\alpha_Y$.

Therefore:

$$C_{N_\alpha}QA\alpha_p\cos\theta = C_{N_\alpha}QA\left[\frac{\dot{X}}{|V|}\sin\theta\cos\theta + \frac{\dot{Y}}{|V|}\psi\sin\theta\cos\theta + \frac{\dot{Z}}{|V|}\cos^2\theta\right]$$

and

$$C_{N_\alpha}QA\alpha_Y = C_{N_\alpha}QA\left[\frac{\dot{X}}{|V|}\psi\cos^2\theta - \frac{\dot{Y}}{|V|} - \frac{\dot{Z}}{|V|}\psi\sin\theta\cos\theta\right]$$

The Angles of Attack in pitch and yaw are defined then as follows:

(32)

$$\alpha_p = \frac{\dot{X}}{|V|}\sin\theta + \frac{\dot{Y}}{|V|}\psi\sin\theta + \frac{\dot{Z}}{|V|}\cos\theta$$

$$\alpha_Y = \frac{\dot{X}}{|V|}\psi\cos^2\theta - \frac{\dot{Y}}{|V|} - \frac{\dot{Z}}{|V|}\psi\sin\theta\cos\theta$$

The rates can be expanded to a standard rate and a perturbation rate as:

$$\dot{X} = \dot{X}_s + \Delta\dot{X}$$

$$\dot{Y} = \dot{Y}_s + \Delta\dot{Y}$$

$$\dot{Z} = \dot{Z}_s + \Delta\dot{Z}$$

also

$$\frac{\dot{X}_s}{|V|} = \sin\Phi_t = \cos\theta_s \qquad \frac{\dot{Z}_s}{|V|} = -\sin\theta_s = -\cos\Phi_t$$

Substituting the above identities into equation 32 gives the following: (where $(\theta-\theta_s)$, $(\Delta\dot{x}/V)$, $(\Delta\dot{Y}/V)$, and $(\Delta\dot{z}/V)$ are treated as small quantities just as is ψ)

(33a)

$$\alpha_p = \theta + \Phi_t - \frac{\pi}{2} + \frac{\Delta\dot{X}\sin\theta + \Delta\dot{Z}\cos\theta}{|V|}$$

(33b) $$\alpha_Y = \psi\cos\theta - \frac{\Delta\dot{Y}}{|V|}$$

Wind effects can be added to the α_p and α_Y equations. The terms in equations 30 that can be further reduced by substituting α_p or α_Y where possible are:

(1) $$\frac{C_{N_\alpha}QA}{m}\left[\frac{\dot{X}}{|V|}\sin^2\theta + \frac{\dot{Y}}{|V|}(\sin^2\theta\cdot\psi - \psi) + \frac{\dot{Z}}{|V|}\cos\theta\sin\theta\right]$$

becomes $$\frac{C_{N_\alpha}QA}{m}\alpha_p\sin\theta$$

(2) $$\frac{C_{N_\alpha}QA}{m}\left[\frac{\dot{X}}{|V|}(\sin^2\theta\cdot\psi - \psi) + \frac{\dot{Y}}{|V|} + \frac{\dot{Z}}{|V|}\cos\theta\sin\theta\cdot\psi\right]$$

becomes $$-\frac{C_{N_\alpha}QA}{m}\alpha_Y$$

(3) $$\frac{C_{N_\alpha}QA}{m}\left[\frac{\dot{X}}{|V|}\sin\theta\cos\theta + \frac{\dot{Y}}{|V|}\psi\sin\theta\cos\theta + \frac{\dot{Z}}{|V|}\cos^2\theta\right]$$

becomes $$\frac{C_{N_\alpha}QA}{m}\alpha_p\cos\theta$$

(4) $$\frac{C_{N_\alpha}QA}{I_{x'}}\left[\frac{\dot X}{|V|}(\cos\phi\sin\theta+\sin\phi\cdot\psi)+\frac{\dot Y}{|V|}(\cos\phi\sin\theta\psi-\sin\phi)+\frac{\dot Z}{|V|}\cos\phi\cos\theta\right]$$

becomes $$\frac{C_{N_\alpha}QA}{I_{x'}}(\alpha_P\cos\phi+\alpha_Y\sin\phi)$$

(5) $$\frac{C_{N_\alpha}QA}{I_{x'}}\left[\frac{\dot X}{|V|}(\sin\phi\sin\theta-\cos\phi\psi)+\frac{\dot Y}{|V|}(\cos\phi+\sin\phi\sin\theta\psi)+\frac{\dot Z}{|V|}\sin\phi\cos\theta\right]$$

becomes $$\frac{C_{N_\alpha}QA}{I_{x'}}(\alpha_P\sin\phi-\alpha_Y\cos\phi)$$

(6) $$\frac{C_{N_\alpha}QA}{I}\left[\frac{\dot X}{|V|}\sin\theta+\frac{\dot Y}{|V|}\psi\sin\theta+\frac{\dot Z}{|V|}\cos\theta\right]$$

becomes $$\frac{C_{N_\alpha}QA}{I}\alpha_P$$

and finally,

(7) $$\frac{C_{N_\alpha}QA}{I}\left[\frac{\dot X}{|V|}\psi-\frac{\dot Y}{|V|}\right]$$ becomes $$\frac{C_{N_\alpha}QA}{I}\alpha_Y$$

Substitution of the above identities back into equations 30 presents the linearized six-degree-of-freedom perturbation equations as:

(34a) $$\Delta\ddot X+\frac{D_e}{m}(\cos\theta-\sin\Phi_t)+\frac{C_{N_\alpha}QA}{m}\alpha_P\sin\theta-\left(\frac{F_I+F_{III}}{m}\right)(\sin\phi\sin\theta-\psi\cos\phi)-\frac{F_T}{m}(\cos\theta-\sin\Phi_t)-\left(\frac{F_{II}+F_{IV}}{m}\right)(\cos\phi\sin\theta+\psi\sin\phi)-\frac{F_T}{m}\beta_Y(\sin\phi\sin\theta-\psi\cos\phi)-\frac{F_T}{m}\beta_P(\cos\phi\sin\theta+\sin\phi\cdot\psi)=0$$

(34b) $$\Delta\ddot Y+\frac{D_e}{m}\psi\cos\theta-\frac{C_{N_\alpha}QA}{m}\alpha_Y-\left(\frac{F_I+F_{III}}{m}\right)(\cos\phi+\sin\phi\sin\theta\psi)-\left(\frac{F_{II}+F_{IV}}{m}\right)(\cos\phi\sin\theta\cdot\psi-\sin\phi)-\frac{F_T}{m}\cos\theta\cdot\psi-\frac{F_T}{m}\beta_Y(\cos\phi+\sin\phi\sin\theta\cdot\psi)-\frac{F_T}{m}\beta_P(\cos\phi\sin\theta\psi-\sin\phi)=0$$

(34c)
$$\Delta\ddot{z} - \frac{D_o}{m}\left[\sin\theta - \cos\Phi_\varepsilon\right] + \frac{C_{N\alpha}QA}{m}\alpha_P\cos\theta$$
$$- \left(\frac{F_I + F_{III}}{m}\right)\sin\phi\cos\theta - \left(\frac{F_{II} + F_{IV}}{m}\right)\cos\phi\cos\theta$$
$$+ \frac{F_T}{m}\left[\sin\theta - \cos\Phi_\varepsilon\right] - \frac{F_T}{m}\beta_Y\cos\theta\sin\phi$$
$$- \frac{F_T}{m}\beta_P\cos\phi\cos\theta = 0$$

(34d)
$$\ddot{\phi} - \ddot{\psi}\sin\theta - \dot{\theta}\dot{\psi}\cos\theta$$
$$+ \frac{C_{N\alpha}QAD}{I_{x'}}\left[\frac{\Delta CP_Y}{D}\left(\alpha_P\cos\phi + \alpha_Y\sin\phi\right) - \frac{\Delta CP_Z}{D}\left(\alpha_P\sin\phi - \alpha_Y\cos\phi\right)\right]$$
$$- \frac{h}{I_{x'}}\left[-F_I + F_{II} + F_{III} - F_{IV}\right] - \frac{\Delta Y_T}{I_{x'}}F_T\beta_P$$
$$+ \frac{\Delta Z_T}{I_{x'}}F_T\beta_Y + \frac{Md_{x'}}{I_{x'}} = 0$$

(34e)
$$\ddot{\theta} + \frac{I_{x'}}{I}\left[\dot{\phi}\dot{\psi}\cos\theta - \dot{\psi}^2\cos\theta\sin\theta\right]$$
$$+ \dot{\psi}^2\cos\theta\sin\theta + D_o\left(\frac{\Delta CP_Z}{I}\cos\phi + \frac{\Delta CP_Y}{I}\sin\phi\right)$$
$$+ \frac{C_{N\alpha}QAD}{I}\left(\frac{CG - CP}{D}\right)\alpha_P$$
$$- \frac{X_c}{I}\left[\cos\phi\cdot\left(F_{II} + F_{IV}\right) + \sin\phi\cdot\left(F_I + F_{III}\right)\right]$$
$$- \frac{X_T}{I}F_T\left(\beta_P\cos\phi + \beta_Y\sin\phi\right) - \frac{\Delta Z_T}{I}F_T\cos\phi$$
$$- \frac{\Delta Y_T}{I}F_T\sin\phi + \frac{Md_{Y'}}{I}\cos\phi - \frac{Md_{Z'}}{I}\sin\phi = 0$$

(34f)
$$\ddot{\psi}\cos\theta - 2\dot{\psi}\dot{\theta}\sin\theta + \frac{I_{x'}}{I}\left[-\dot{\phi}\dot{\theta} + \dot{\psi}\dot{\theta}\sin\theta\right]$$
$$+ D_0\left[\frac{\Delta CP_z}{I}\sin\phi - \frac{\Delta CP_Y}{I}\cos\phi\right] + \frac{C_{N\alpha}QAD}{I}\left(\frac{CG-CP}{D}\right)\alpha_Y$$
$$+ \frac{X_c}{I}\left[(F_I + F_{III})\cos\phi - (F_{II} + F_{IV})\sin\phi\right]$$
$$- \frac{\Delta Z_T}{I}F_T\sin\phi + \frac{\Delta Y_T}{I}F_T\cos\phi + \frac{X_T}{I}F_T\beta_Y\cos\phi$$
$$- \frac{X_T}{I}F_T\beta_P\sin\phi + \frac{M_{dz'}}{I}\cos\phi + \frac{M_{dY'}}{I}\sin\phi = 0$$

(34g)
$$\alpha_P = \theta + \Phi_t - \frac{\pi}{2} + \frac{\Delta\dot{X}\sin\theta + \Delta\dot{Z}\cos\theta}{|V|}$$
$$\alpha_Y = \psi\cos\theta - \frac{\Delta\dot{Y}}{|V|}$$

Section IV - Body Moment Derivation B

The body moments as expressed in equations 3 can be equated to the respective applied moments of equations 17 to give the following equations:

Body Roll Moment

(35a)
$$\dot{p}I_{x'} + (rp - \dot{q})J_{x'y'} - (\dot{r} + pq)J_{z'x'} + (r^2 - q^2)J_{y'z'} + qr(I_{z'} - I_{y'})$$
$$= \frac{C_N QAD}{|\sin\alpha|}\left\{\frac{\Delta CP_Y}{D}\left[-\frac{\dot{X}}{|V|}(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)\right.\right.$$
$$- \frac{\dot{Y}}{|V|}(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)$$
$$\left. - \frac{\dot{Z}}{|V|}\cos\phi\cos\theta\right]$$
$$+ \frac{\Delta CP_z}{D}\left[\frac{\dot{X}}{|V|}(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi)\right.$$
$$+ \frac{\dot{Y}}{|V|}(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi)$$
$$\left.\left. + \frac{\dot{Z}}{|V|}\sin\phi\cos\theta\right]\right\}$$
$$+ h\left[-F_I + F_{II} + F_{III} - F_{IV}\right] + \Delta Y_T F_T\sin\beta_P$$
$$- \Delta Z_T F_T\cos\beta_P\sin\beta_Y - M_{dx'}$$

Body Pitch Moment

(35b)
$$\dot{q} I_{Y'} + (pq - \dot{r}) J_{Y'Z'} - (\dot{p} + qr) J_{X'Y'} + (p^2 - r^2) J_{Z'X'} + rp(I_{X'} - I_{Z'})$$

$$= -D_0 \Delta CP_Z + X_C \left[F_{I} + F_{III} \right] + \Delta Z_T F_T \cos\beta_P \cos\beta_Y$$

$$- \frac{C_N QAD}{|\sin\alpha|} \left(\frac{CG - CP}{D} \right) \left[\frac{\dot{X}}{|V|} (\cos\phi \sin\theta \cos\psi + \sin\phi \sin\psi) \right.$$

$$+ \frac{\dot{Y}}{|V|} (\cos\phi \sin\theta \sin\psi - \sin\phi \cos\psi)$$

$$\left. + \frac{\dot{Z}}{|V|} \cos\phi \cos\theta \right]$$

$$+ X_T F_T \sin\beta_P - Md_{Y'}$$

Body Yaw Moment

(35c)
$$\dot{r} I_{Z'} + (qr - \dot{p}) J_{X'Z'} - (\dot{q} + rp) J_{Y'Z'} + (q^2 - p^2) J_{X'Y'} + pq(I_{Y'} - I_{X'})$$

$$= D_0 \Delta CP_Y - X_C \left[F_{I} + F_{III} \right] - X_T F_T \cos\beta_P \sin\beta_Y$$

$$+ \frac{C_N QAD}{|\sin\alpha|} \cdot \left(\frac{CG - CP}{D} \right) \left[\frac{\dot{X}}{|V|} (\sin\phi \sin\theta \cos\psi - \cos\phi \sin\psi) \right.$$

$$+ \frac{\dot{Y}}{|V|} (\cos\phi \cos\psi + \sin\phi \sin\theta \sin\psi)$$

$$\left. + \frac{\dot{Z}}{|V|} \sin\phi \cos\theta \right]$$

$$- \Delta Y_T F_T \cos\beta_P \cos\beta_Y - Md_{Z'}$$

When the corresponding standard trajectory equations are subtracted from those in equations 35, the perturbation equations of the body moments are obtained.

The perturbation equations of body moments are simplified by incorporating the same assumptions as in derivation A.

The resulting perturbation equations are:

Body roll moment

(36a)
$$\dot{p} I_x = - C_{N_\alpha} Q A D \left\{ \frac{\Delta CP_Y}{D}\left(\alpha_P \cos\phi + \alpha_Y \sin\phi\right) - \frac{\Delta CP_Z}{D}\left(\alpha_P \sin\phi - \alpha_Y \cos\phi\right) \right\} + h\left[-F_{I} + F_{II} + F_{III} - F_{IV}\right] + \Delta Y_T F_T \beta_P - \Delta Z_T F_T \beta_Y - M_{d_{X'}}$$

Body pitch moment

(36b)
$$\dot{q} I + r p \left(I_{X'} - I\right) = -D_0 \Delta CP_Z + X_c\left[F_{II} + F_{IV}\right] - C_{N_\alpha} Q A D \left(\frac{CG - CP}{D}\right)\left(\alpha_P \cos\phi + \alpha_Y \sin\phi\right) + F_T \Delta Z_T + X_T F_T \beta_P - M_{d_{Y'}}$$

Body Yaw Moment

(36c)
$$\dot{r} I_{z'} + pq\left(I_{Y'} - I_{X'}\right) = D_0 \Delta CP_Y - X_c\left[F_I + F_{III}\right] - X_T F_T \beta_Y - \Delta Y_T F_T - M d_{z'} + C_{N\alpha} Q A D\left(\frac{CG-CP}{D}\right)\left[\alpha_P \sin\phi - \alpha_Y \sin\phi\right]$$

where α_p and α_Y have been defined previously as:

$$\alpha_p = \theta + \Phi_t - \frac{\pi}{2} + \frac{\Delta\dot{x}\sin\theta + \Delta\dot{z}\cos\theta}{|V|}$$

$$\alpha_Y = \psi\cos\theta - \frac{\Delta\dot{Y}}{|V|}$$

$$|V| = +\sqrt{\bar{V}\cdot\bar{V}}$$

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1. Airplane Performance Stability and Control by Courtland D. Perkins and Robert E. Hage published by John Wiley & Sons, Inc. copyright 1949.

2. Ballistics of the Future by Dr. J.M.J. Kooy and Prof. IR J.W.H. Uytenbogaart published by McGraw Hill Book Company, Inc. copyright 1946.

3. AME-TN-7-60 Non-Linear Six-Degree of Freedom Equations of Motion for a Ballistic Missile by R.J. Thompson and G.A. Socks, 16 March 1960.

Nomenclature

Symbol	Definition	Units
A	Missile reference area	M^2
C_{N_α}	Gradient of the normal force coefficient	1/RAD
C_N	Normal force coefficient	
D_o	Axial drag term	kg
D	Reference diameter of missile	M
F_N	Aerodynamic normal force	Kg
F_{aero}	Total aerodynamic force	Kg
F_c	Thrust of each control member	Kg
F_T	Engine thrust vector	Kg
$F(X, Y, Z)$	Force with components in X, Y, Z space axes system	Kg
CG-CP	Axial distance between the center of gravity and the center of pressure of the body	M
g	Acceleration due to gravity	M/sec^2
h	Distance of control force from body fixed longitudinal axis	M
$h_{x'}$	Moment of momentum about body roll axis	Kg M sec
$h_{y'}$	Moment of momentum about body pitch axis	Kg M sec
$h_{z'}$	Moment of momentum about body yaw axis	Kg M sec
$I_{x'}$	Moment of inertia about body roll axis	Kg M sec^2
$I_{y'}$	Moment of inertia about body pitch axis	Kg M sec^2
$I_{z'}$	Moment of inertia about body yaw axis	Kg M sec^2
$J_{x'z'}$	Product of inertia about body pitch axis	Kg M sec^2
$J_{x'y'}$	Product of inertia about body yaw axis	Kg M sec^2
$J_{y'z'}$	Product of inertia about body roll axis	Kg M sec^2
p	Body roll rate	rad/sec

q	Body pitch rate	rad/sec
r	Body yaw rate	rad/sec
M	Moment	Kg M
M_{aero}	Total aerodynamic body moment	Kg M
M_{CF}	Moment due to the control system	Kg M
M_{Th}	Moment due to thrust vector misalignment	Kg M
Mg	Missile weight vector	Kg
m	Mass of the missile	Kg sec^2/M
Md_x	Damping moment along body roll axis	Kg M
Md_y	Damping moment along body pitch axis	Kg M
Md_z	Damping moment along body yaw axis	Kg M
Q	Dynamic pressure	Kg/M^2
R_o	Mean radius of earth	M
R	Distance from the center of the earth to the missile CG.	M
R_{in}	Initial distance from the center of the earth to the missile CG.	M
r_T	Distance from body center of gravity to application of thrust vector	M
i	Unit vector along space fixed X axis	
j	Unit vector along space fixed Y axis	
k	Unit vector along space fixed Z axis	
i'	Unit vector along body fixed X axis	
j'	Unit vector along body fixed Y axis	
k'	Unit vector along body fixed Z axis	
ψ, θ, ϕ	Eulerian angles of rotation of the body axes around the X, Y, Z space fixed axes system	rad
Ω	Angle at the earth's center between initial local vertical and the local vertical at any time	rad

β_Y , β_P	Thrust vector misalignment angles (see diagram 2)	rad.
θ_S	Standard trajectory Euler pitch angle	rad.
α	Total angle of attack	rad.
α_P , α_Y	Components of α in the pitch and yaw planes respectively	rad.
Φ_T	Angle between the original local vertical and the velocity vector	rad.
X_T	Distance from body C.G. to application of thrust vector parallel to the longitudinal body axis.	M
X_c	Distance along the body longitudinal axis to the control force station from the body CG.	M
V	Missile velocity	M/sec
$\dot{X}_s$, $\dot{Y}_s$, $\dot{Z}_s$	Space fixed standard trajectory velocities	M/sec
X', Y', Z'	Body Eulerian axes system	
X, Y, Z	Space fixed axes system	
ΔY_T, ΔZ_T	Displacement of the thrust vector in the body Y' and Z' directions respectively (see diagram 2)	M
CP_y	Displacement of the center of pressure on the Y axis	M
CP_z	Displacement of the center of pressure on the Z axis	M

Moments of Momentum Referred to Moving Axis

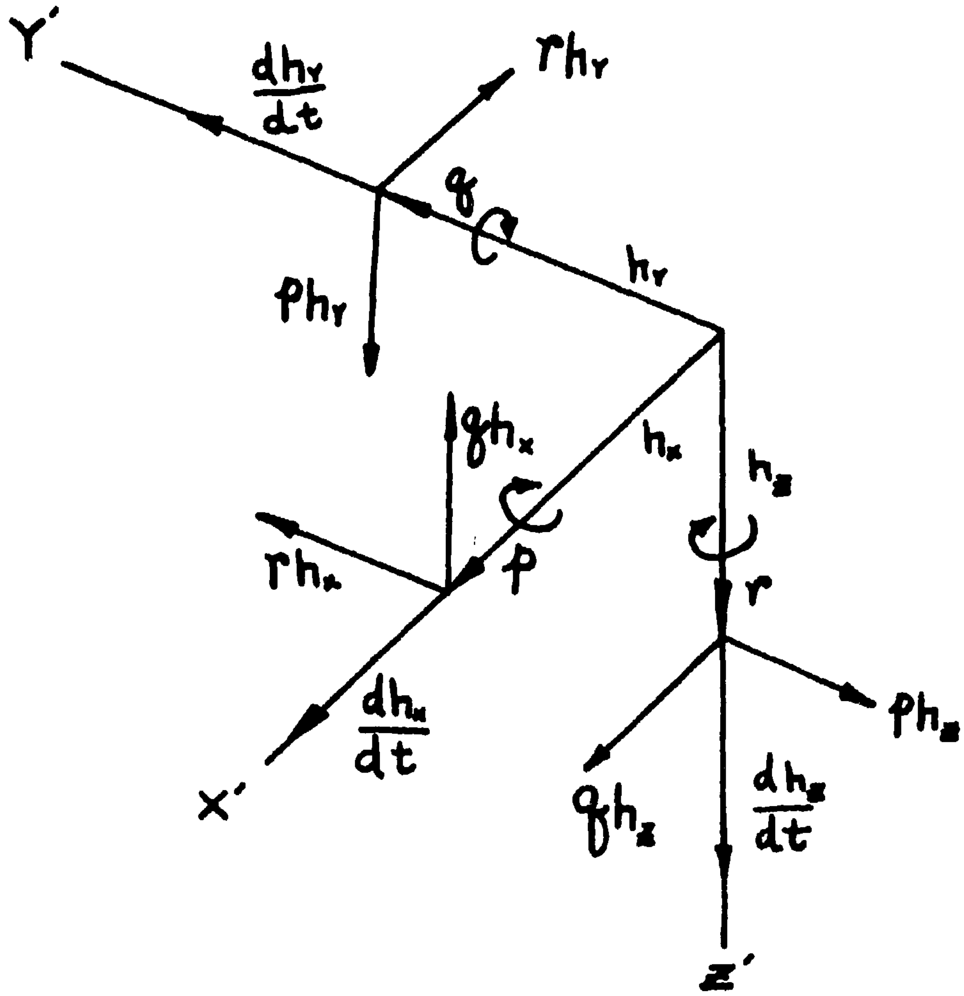
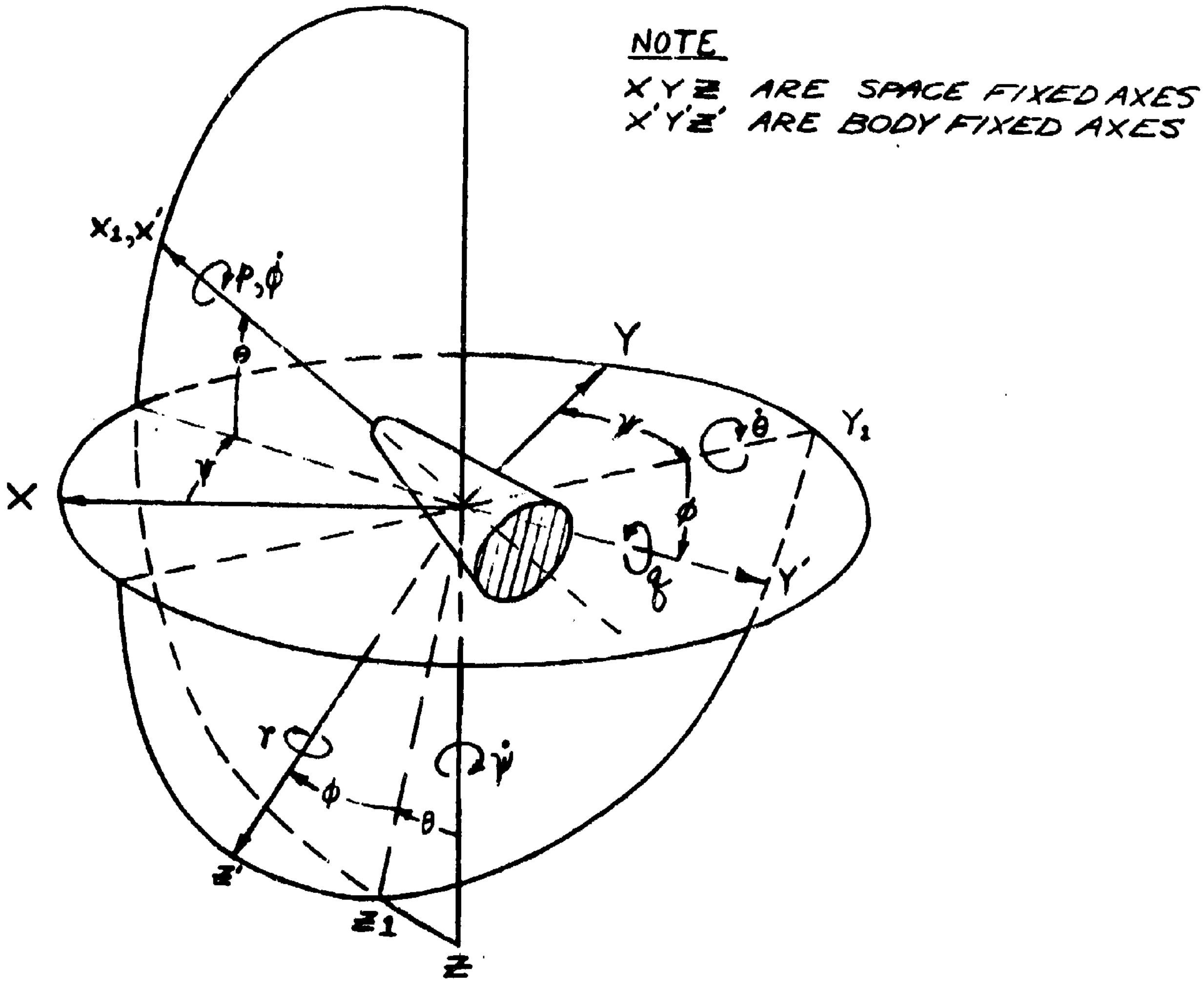


Figure 1

Coordinate System

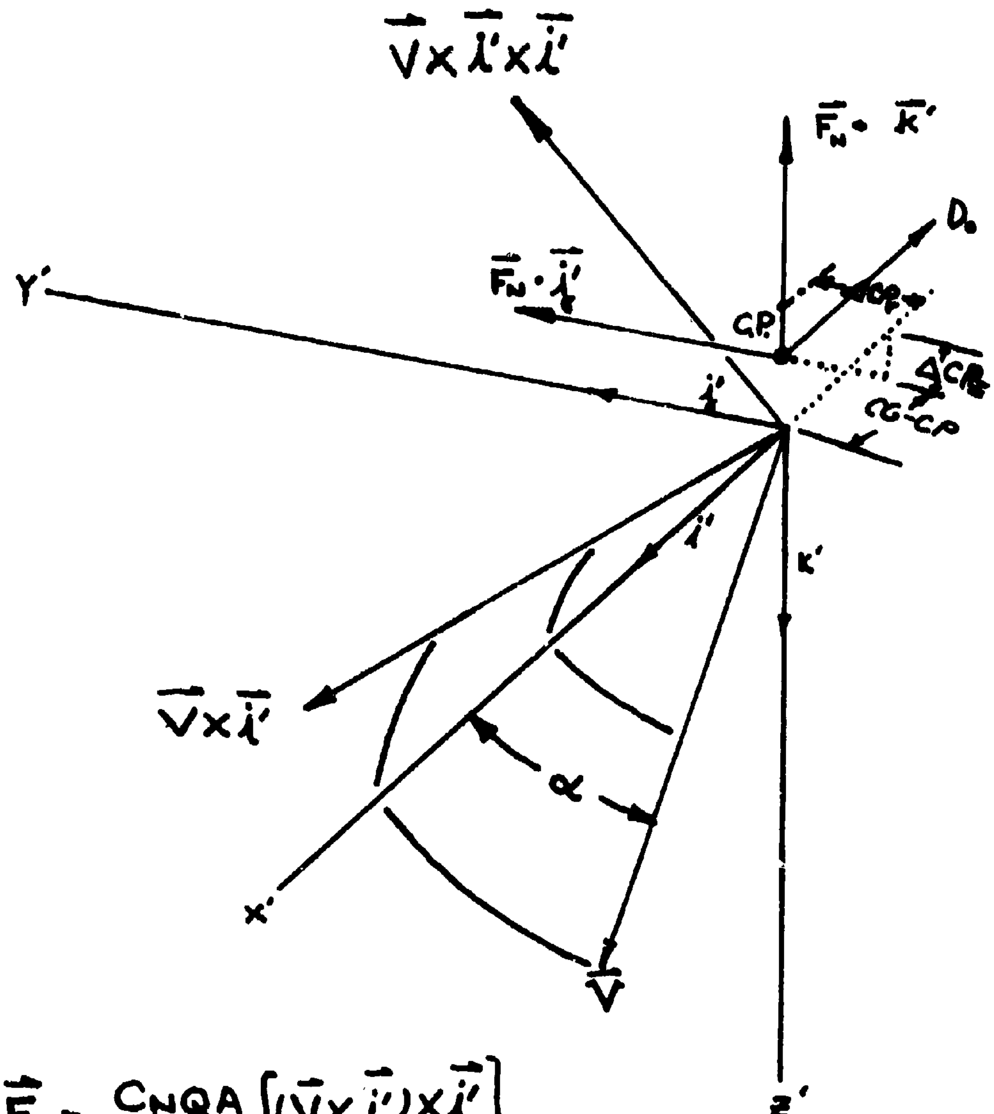


$$\vec{V}_{\text{BODY AXES}} = \left[T_{SB}\right] \vec{V}_{\text{SPACE FIXED AXES}}$$

$$\left[T_{SB}\right] = \begin{bmatrix} \cos\psi\cos\theta & \sin\psi\cos\theta & -\sin\theta \\ \cos\psi\sin\theta\sin\phi - \cos\phi\sin\psi & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & \sin\phi\cos\theta \\ \sin\phi\sin\psi + \cos\psi\sin\theta\cos\phi & \sin\psi\sin\theta\cos\phi - \sin\phi\cos\psi & \cos\theta\cos\phi \end{bmatrix}$$

Figure 2

Aerodynamic Force



$$\vec{F}_N = \frac{C_N Q A}{|V \sin\alpha|}\left[(\vec{V} \times \vec{i}') \times \vec{i}'\right]$$

Figure 3

Weight Vector Orientation

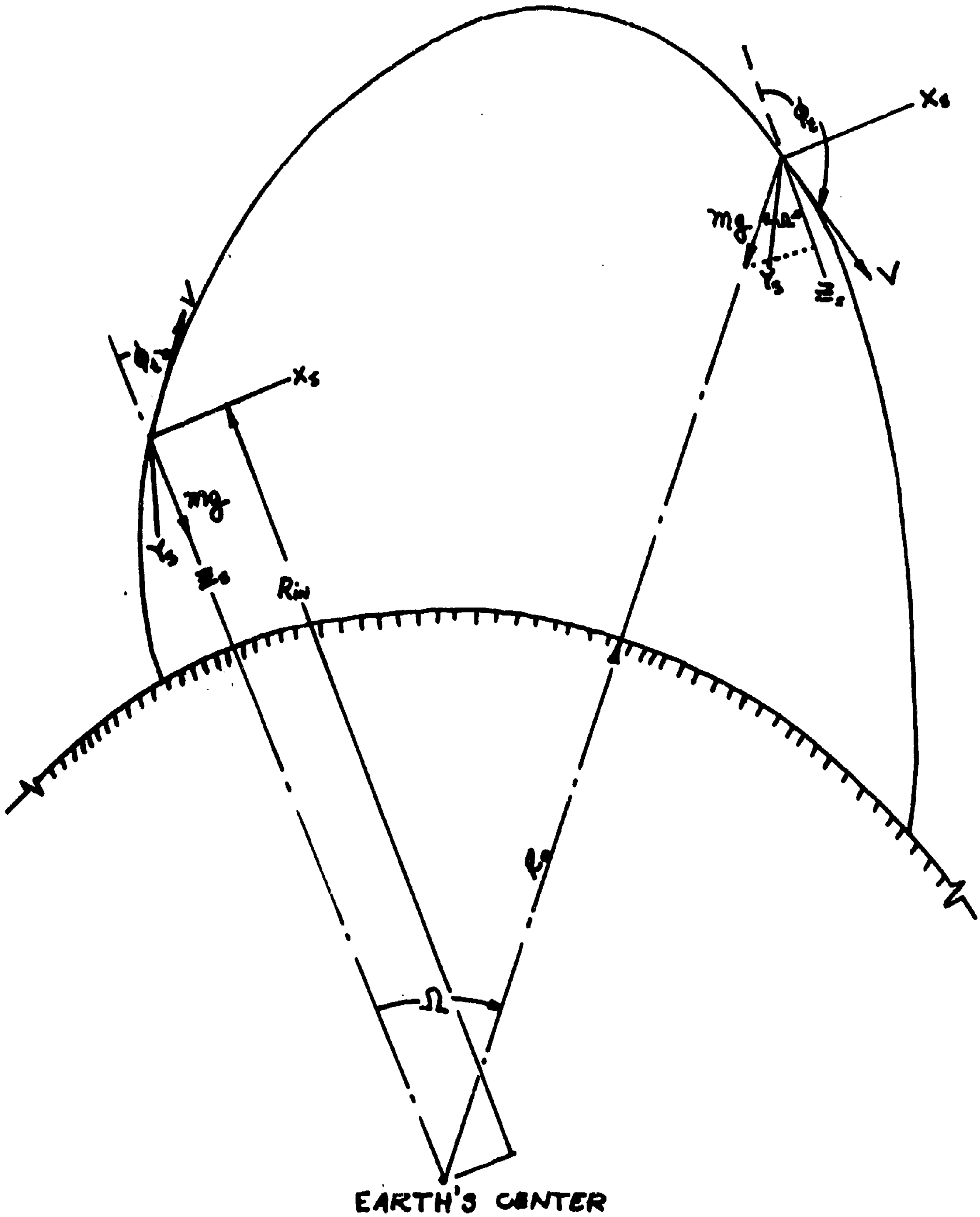


Figure 4

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